The Price of Anarchy in Auctions

Part II: The Smoothness Framework

Jason Hartline
Northwestern University

Vasilis Syrgkanis
Cornell University

December 11, 2013

Part II: High-level goals

- PoA in auctions (as games of incomplete information):
  - Single-Item First Price, All-Pay, Second Price Auctions
  - Simultaneous Single Item Auctions
  - Position Auctions: GSP, GFP
  - Combinatorial auctions

General Approach

- Reduce analysis of complex setting to simple setting.
- Conclusion for simple setting X, proved under restriction P, extends to complex setting Y
  - X: complete information PNE to Y: incomplete information BNE
  - X: single auction to Y: composition of auctions

Best-Response Analysis

- Objective in X is good because each player doesn't want to deviate to strategy b_i'

- Extension from setting X to setting Y: if best response argument satisfies condition P then conclusion extends to Y

5

First Extension Theorem

Complete info PNE to BNE with correlated values

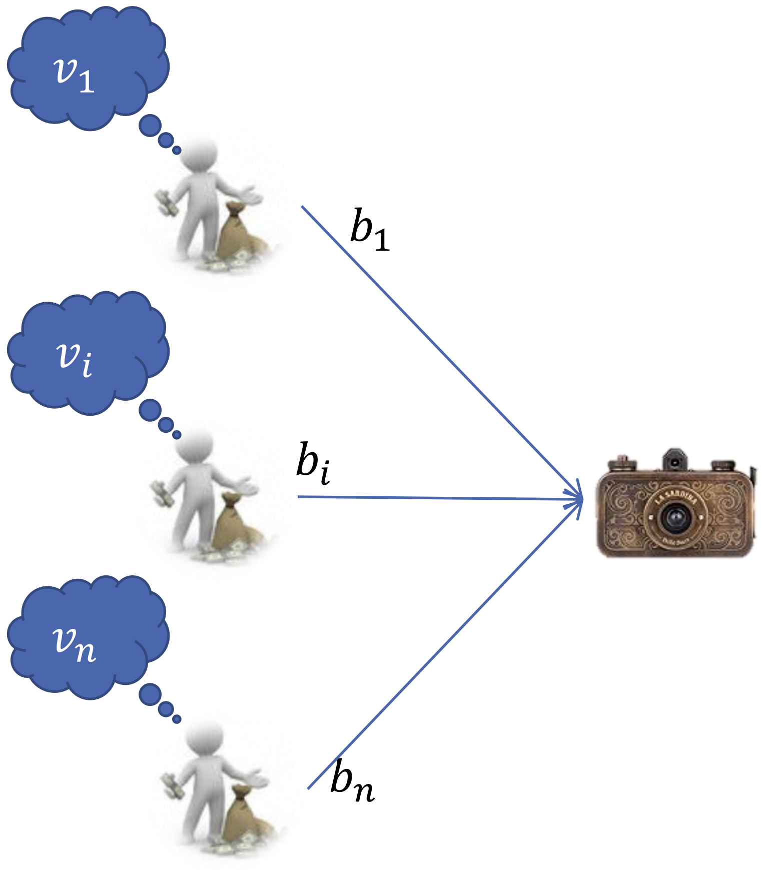
First Extension Theorem

Complete info PNE to BNE with correlated values

- Target setting. First Price Bayes-Nash Equilibrium with asymmetric correlated values
- Simple setting. Complete information Pure Nash Equilibrium
- **Thm.** If proof of PNE PoA based on own-value based deviation argument then PoA of BNE also good

References:
Roughgarden STOC'09
Lucier, Paes Leme EC'11
Roughgarden EC'12
Syrgkanis '12
Syrgkanis, Tardos STOC'13

First-Price Auction Refresher



- Highest bidder wins:

$$x_i(\mathbf{b}) = \{\textit{indicator that } i \textit{ wins}\}$$

- Pays his bid: $P_i(\mathbf{b}) = b_i \cdot x_i(\mathbf{b})$
- Quasi-Linear preferences:

$$\text{UTILITY} = \text{VALUE} \ - \text{PAYMENT}$$

$$u_i(\mathbf{b}) = (v_i - b_i) \cdot x_i(\mathbf{b})$$

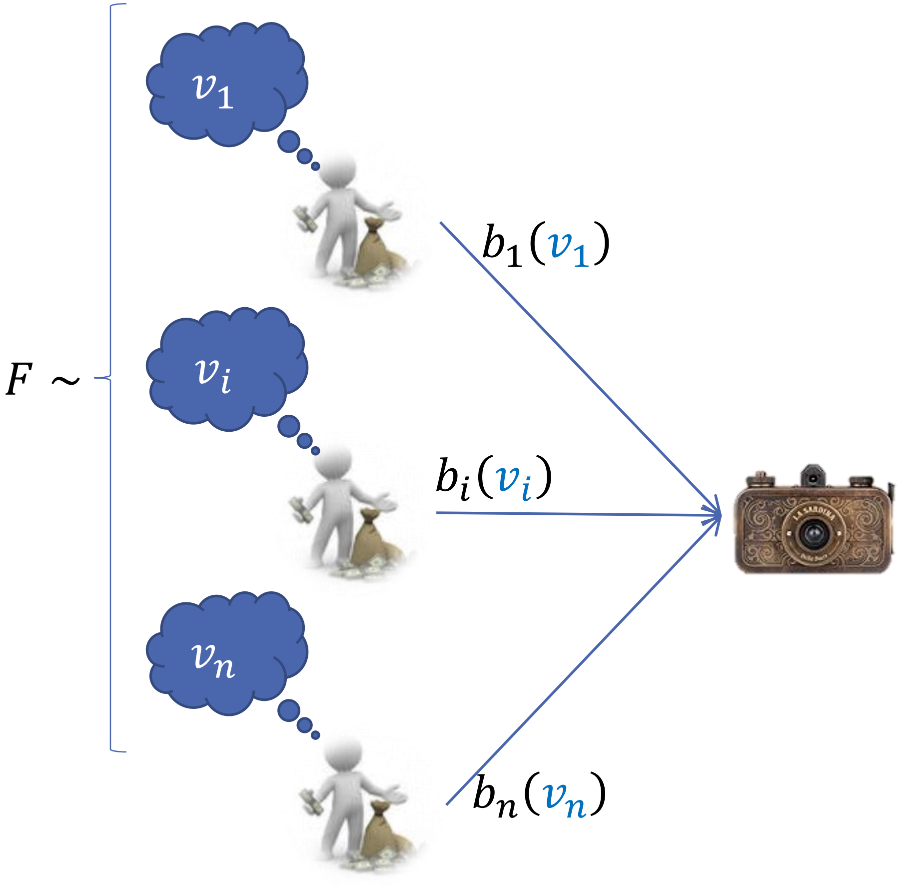
- Objective:

$$\text{WELFARE} = \text{UTILITIES} + \text{PAYMENTS}$$

$$SW(\mathbf{b}) = \sum_i u_i(\mathbf{b}) + \sum_i P_i(\mathbf{b})$$

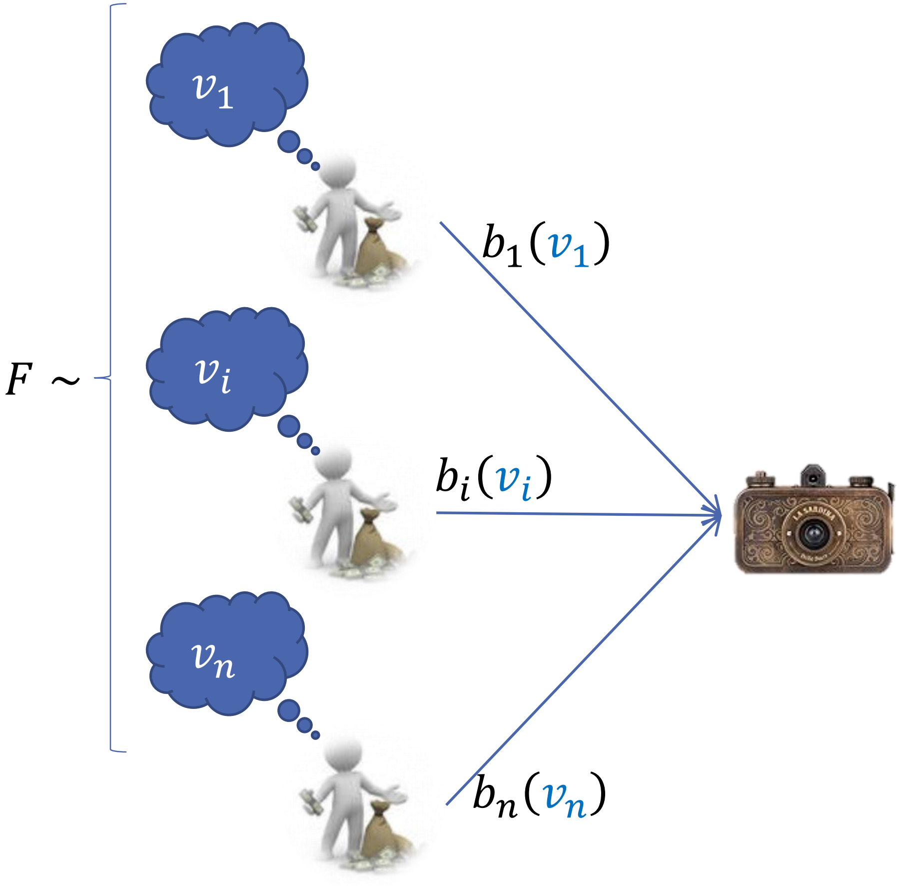
$$= \sum_i \big(u_i(\mathbf{b}) + b_i \cdot x_i(\mathbf{b})\big) = \sum_i v_i \cdot x_i(\mathbf{b})$$

First-Price Auction
Target: BNE with correlated values



- $\mathbf{v} = (v_1, \dots, v_n) \sim F$: correlated distribution
- Conditional on value, maximizes utility:
$$E\left[u_i\left(\mathbf{b}(\mathbf{v})\right) \mid v_i\right] \geq E\left[u_i\left(b_i', \mathbf{b}_{-\mathbf{i}}(\mathbf{v}_{-\mathbf{i}})\right) \mid v_i\right]$$
- Equilibrium Welfare:
$$E\left[SW\left(\mathbf{b}(\mathbf{v})\right)\right] = E\left[\sum_i v_i \cdot x_i\left(\mathbf{b}(\mathbf{v})\right)\right]$$
- Optimal Welfare: highest value bidder
$$E[OPT(\mathbf{v})] = E\left[\sum_i v_i \cdot x_i^*(\mathbf{v})\right]$$

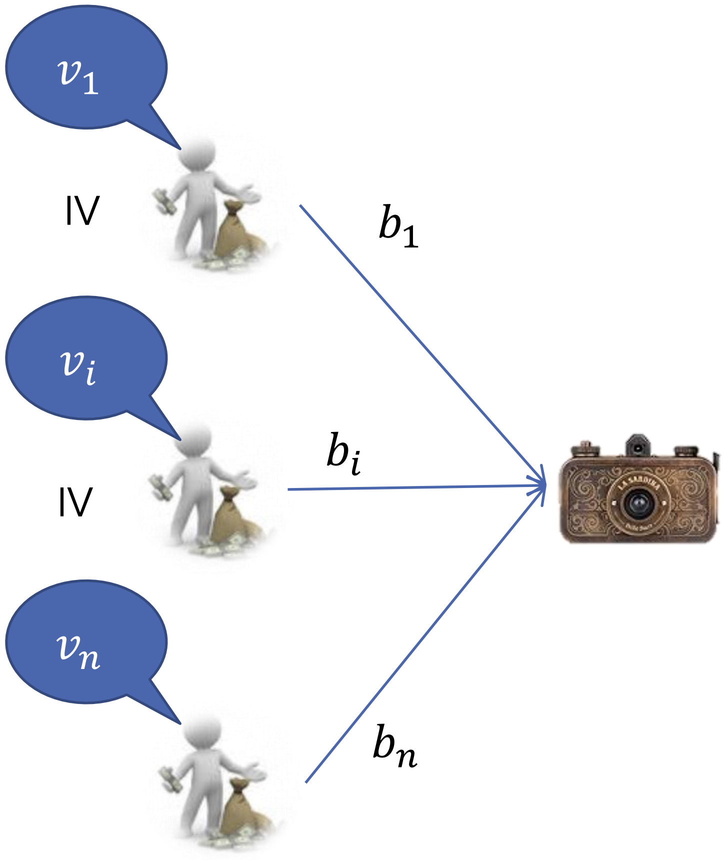
First-Price Auction
Target: BNE with correlated values



$$PoA = \frac{E[OPT(\mathbf{v})]}{E[SW(\mathbf{b}(\mathbf{v}))]}$$

First-Price Auction

Simpler: PNE and complete Information



- $v = (v_1, \dots, v_n)$: common knowledge

- b_i maximizes utility:

$$u_i(b) \geq u_i(b_i', b_{-i})$$

- Equilibrium Welfare:

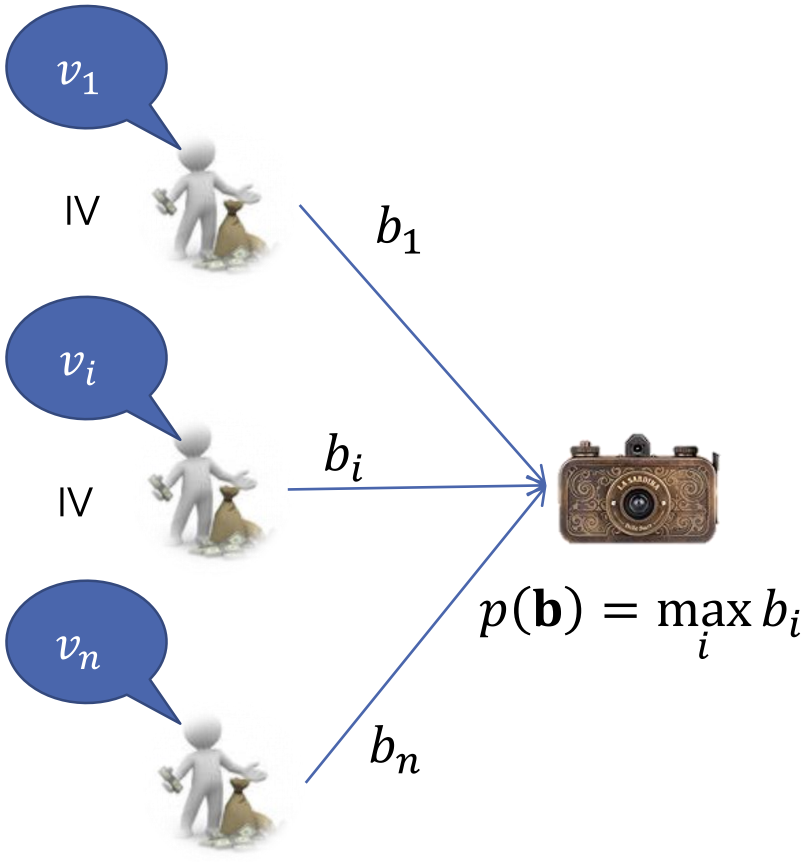
$$SW(b) = \sum_i v_i \cdot x_i(\mathbf{b})$$

- Optimal Welfare:

$$OPT(v) = \sum_i v_i \cdot x_i^*(\mathbf{v})$$

First-Price Auction

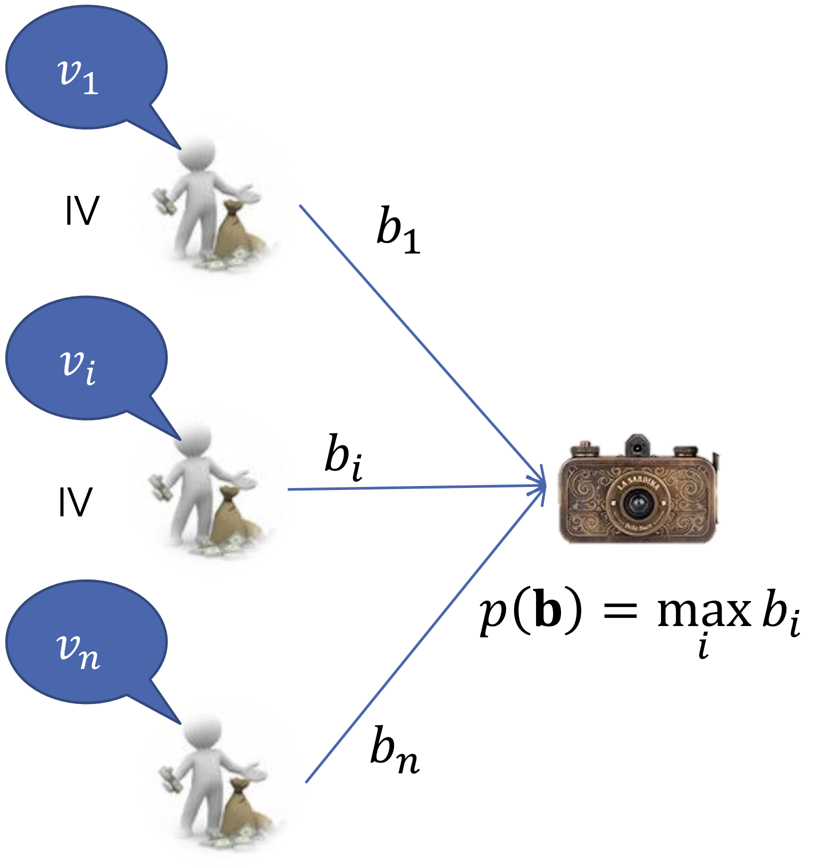
Simpler: PNE and complete Information



$$PoA = \frac{OPT(\mathbf{v})}{SW(\mathbf{b})}$$

First-Price Auction

Simpler: PNE and complete Information



Theorem. $PoA = 1$

Proof. Highest value player can deviate to $p(\mathbf{b})^+$

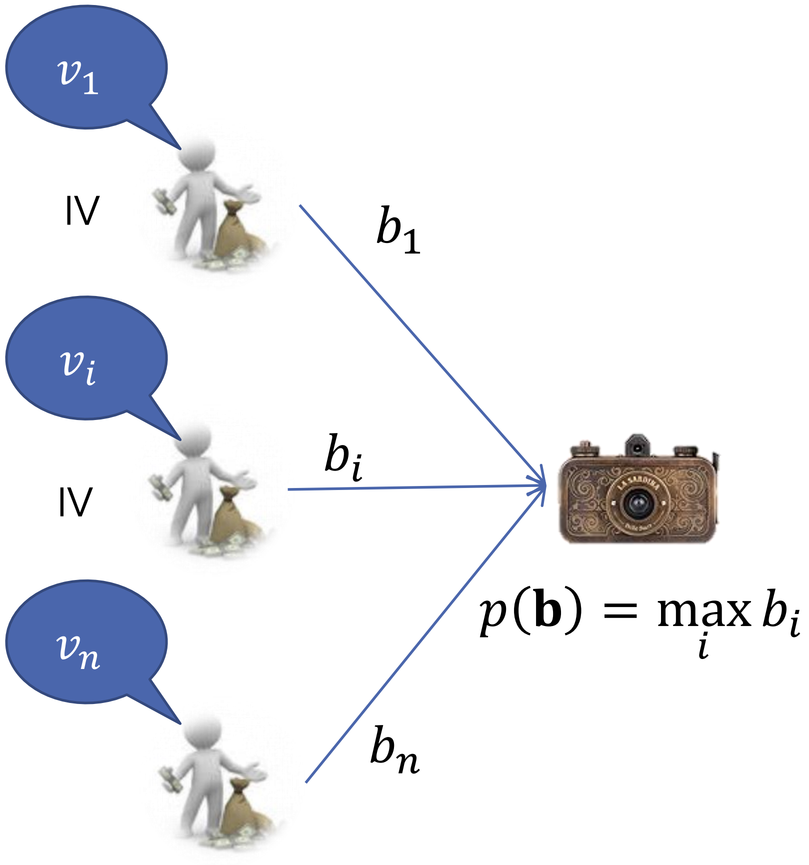
$$u_1(p(\mathbf{b})^+, \mathbf{b_{-i}}) = v_1 - p(\mathbf{b})^+$$
$$u_i(0, \mathbf{b_{-i}}) = 0$$

$$\sum_i u_i(\mathbf{b}) \geq \sum_i u_i(b_i', \mathbf{b_{-i}}) = v_1 - p(\mathbf{b})$$

By PNE condition

First-Price Auction

Simpler: PNE and complete Information



Theorem. $PoA = 1$

Proof. Highest value player can deviate to $p(\mathbf{b})^+$

$$u_1(p(\mathbf{b})^+, \mathbf{b_{-i}}) = v_1 - p(\mathbf{b})^+$$
$$u_i(0, \mathbf{b_{-i}}) = 0$$

$$UTIL(b) \geq \sum_i u_i(b_i', \mathbf{b_{-i}}) = v_1 - REV(b)$$

$$UTIL(b) + REV(b) \geq v_1$$

$$SW(b) \geq v_1$$

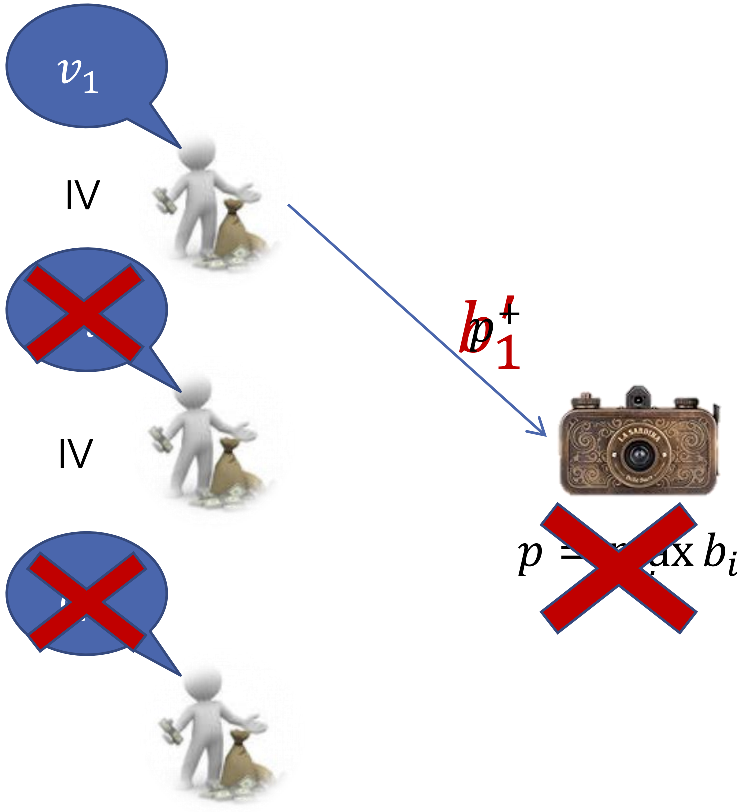
Direct extensions

- What if conclusions for PNE of complete information directly extended to:
  - incomplete information BNE
  - simultaneous composition of single-item auctions

- Obviously: $PoA = 1$ doesn't carry over

- Possible, but we need to restrict the type of analysis

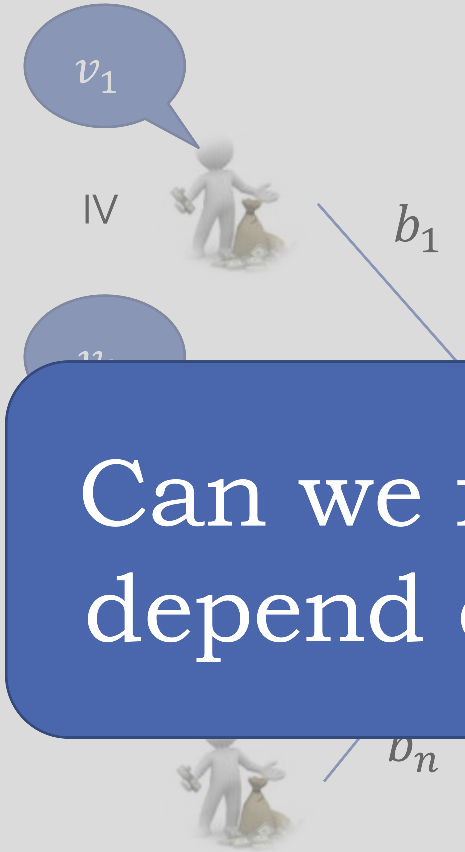
Problem in previous PNE proof



- Recall. $PoA = 1$ because highest value player doesn't want to deviate to p^+
- Challenge. Don't know p or $\mathbf{v_{-i}}$ in incomplete information
- Idea. Restrict deviation to not depend on these parameters

First-Price Auction

Simpler: PNE a



Recall PoA=1 Proof

Proof. Highest value player can deviate to $p(\mathbf{b})^+$

$$u_1(p(\mathbf{b})^+, \mathbf{b_{-i}}) = v_1 - p(\mathbf{b})^+$$
$$u_i(0, \mathbf{b_{-i}}) = 0$$

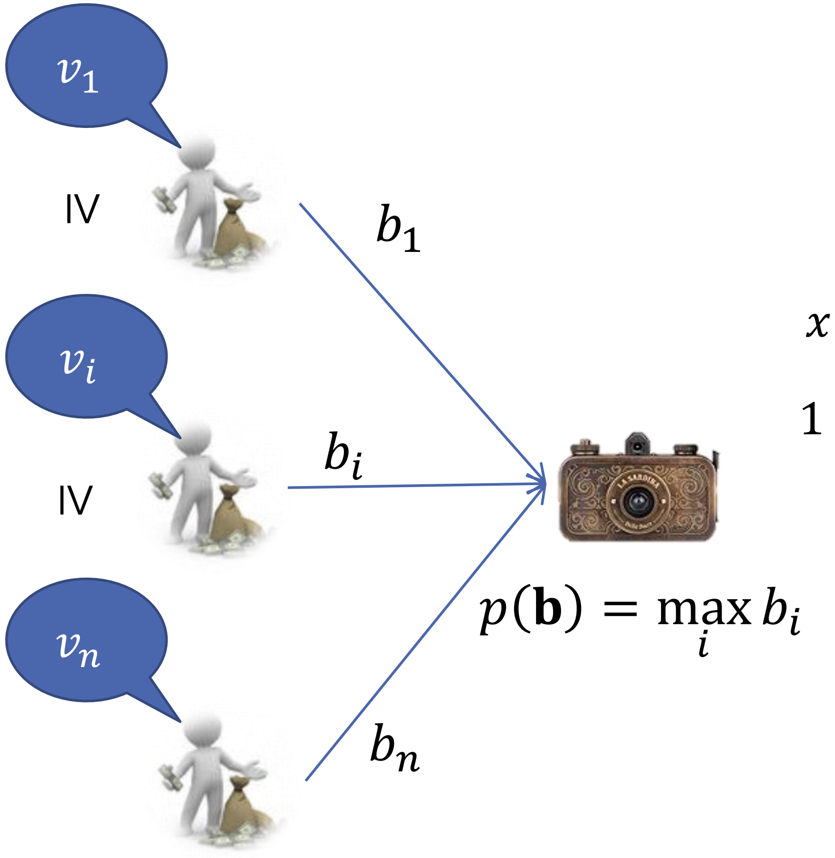
$$U(b) \geq \sum_i u_i(b_i', \mathbf{b_{-i}}) = v_1 - REV(b)$$

$$U(b) + REV(b) \geq v_1$$

$$SW(b) \geq v_1$$

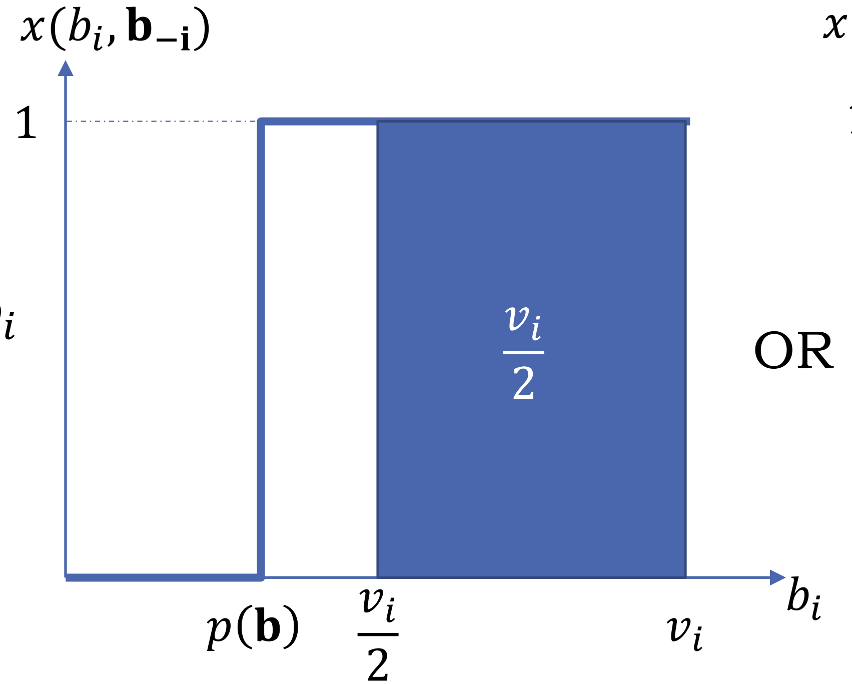
Can we find b_i' that depend only on v_i?

Own-value deviations
(price and other values oblivious)

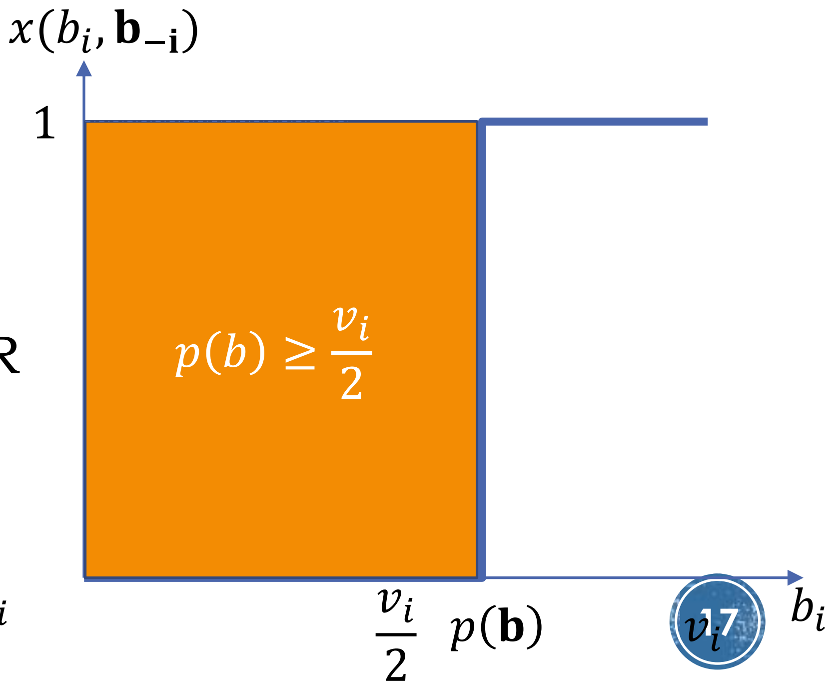


New Theorem. $PoA \leq \mathbf{2}$

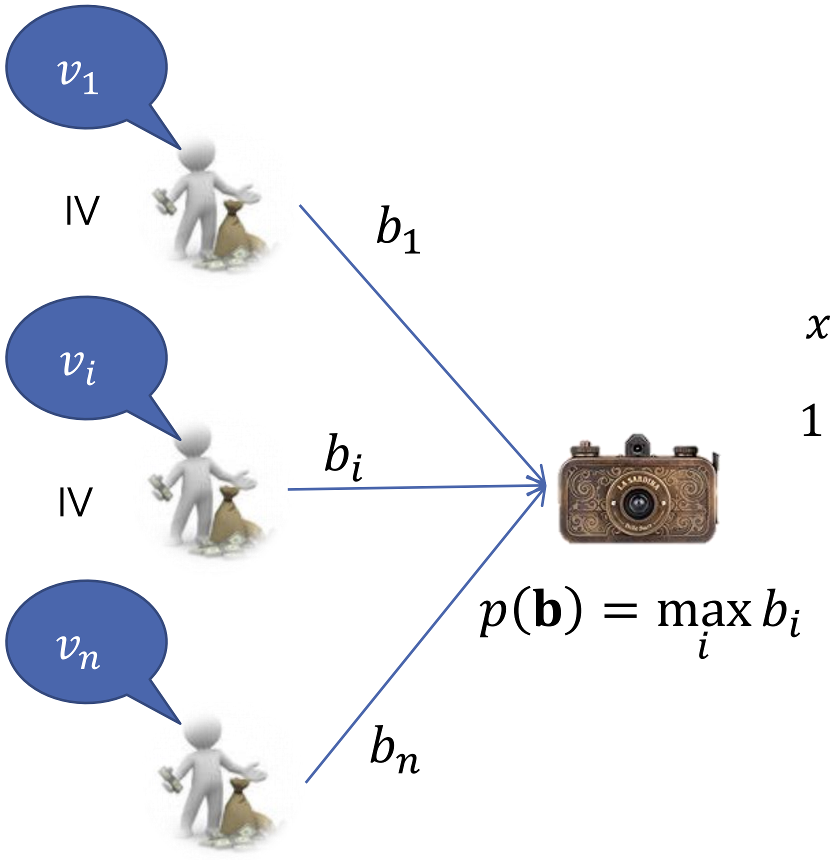
Proof. Each player can deviate to $b_i' = \frac{v_i}{2}$



OR

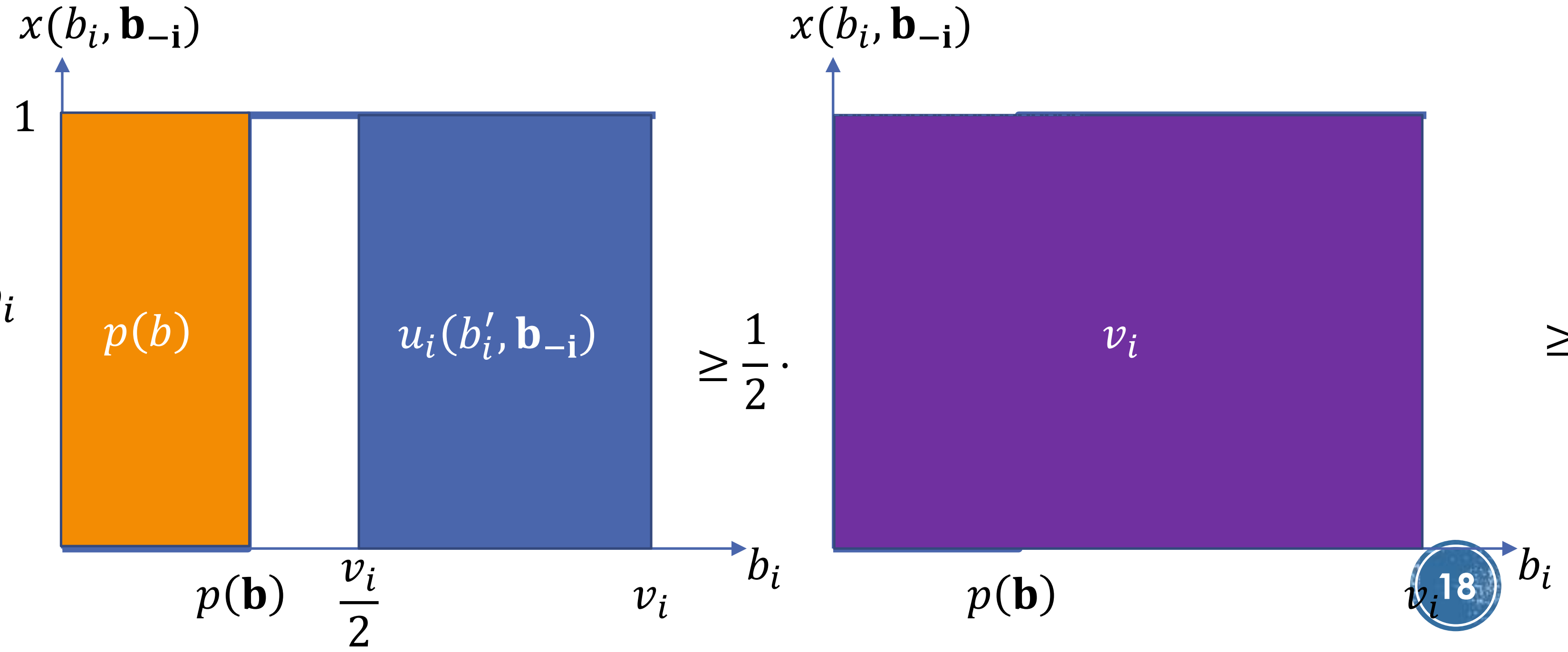


Own-value deviations
(price and other values oblivious)

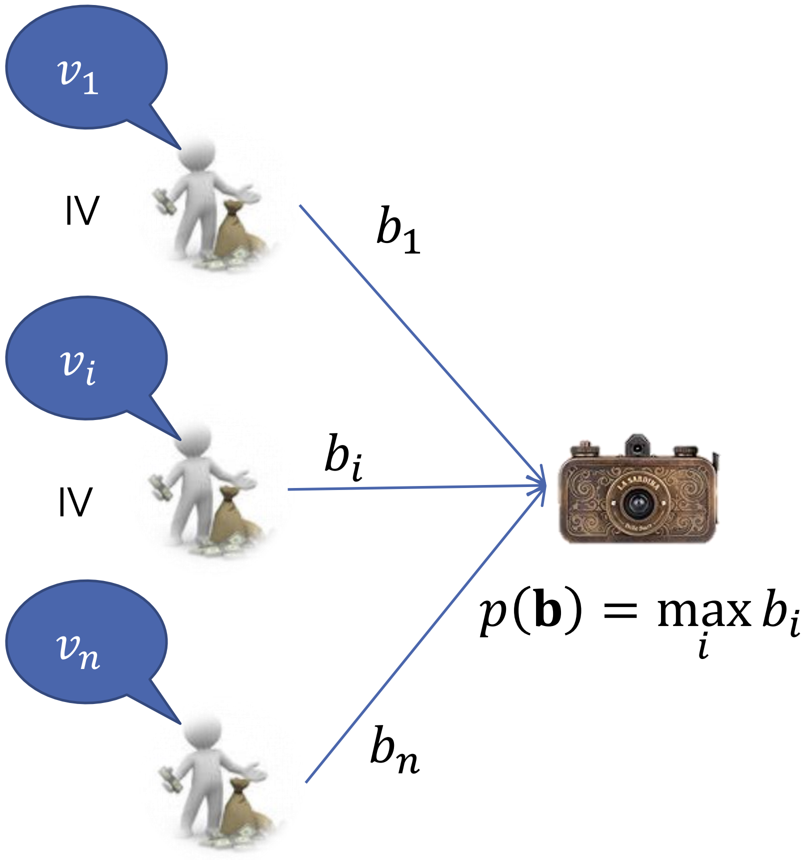


New Theorem. $PoA \leq 2$

Proof. Each player can deviate to $b_i' = \frac{v_i}{2}$



Own-value deviations
(price and other values oblivious)

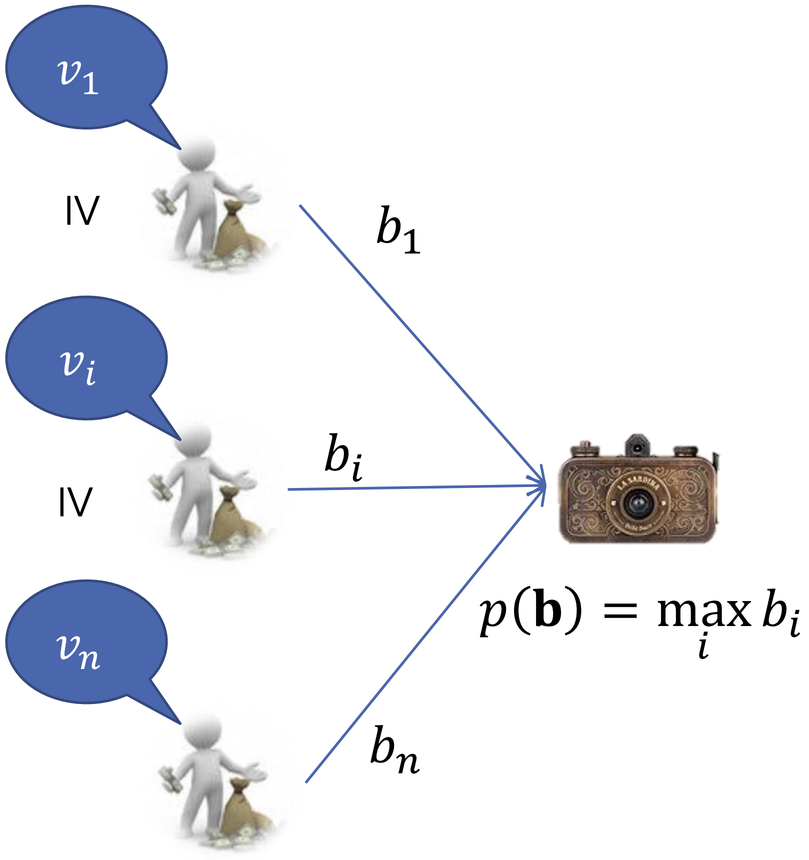


New Theorem. $PoA \leq \mathbf{2}$

Proof. Each player can deviate to $b_i' = \frac{v_i}{2}$

$$u_i\left(\frac{v_i}{2}, \mathbf{b_{-i}}\right) + p(\mathbf{b}) \geq \frac{v_i}{2}$$

Own-value deviations
(price and other values oblivious)



New Theorem. $PoA \leq \mathbf{2}$

Proof. Each player can deviate to $b_i' = \frac{v_i}{2}$

$$u_i\left(\frac{v_i}{2}, \mathbf{b_{-i}}\right) + p(\mathbf{b}) \cdot x_i^*(\mathbf{v}) \geq \frac{v_i}{2} \cdot x_i^*(\mathbf{v})$$

$$UTIL(\mathbf{b}) \geq \sum_i u_i\left(\frac{v_i}{2}, \mathbf{b_{-i}}\right) + p(\mathbf{b}) \geq \frac{1}{2} OPT(\mathbf{v})$$

$$UTIL(\mathbf{b}) + REV(\mathbf{b}) \geq \frac{1}{2} OPT(\mathbf{v})$$

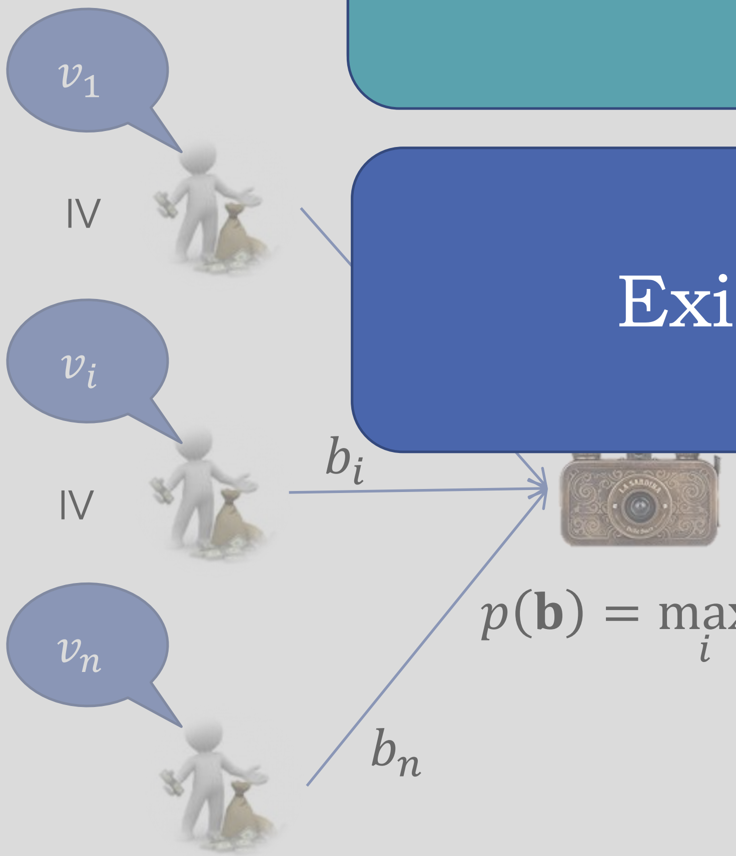
$$SW(\mathbf{b}) \geq \frac{1}{2} OPT(\mathbf{v})$$

Own-value deviations

(pric

Smoothness Property

Exists b_i' depending only on own value



$$p(\mathbf{b}) = \max_i b_i$$

$$UTIL(\mathbf{b}) \geq \sum_i u_i(b_i', \mathbf{b_{-i}}) + REV(\mathbf{b}) \geq \frac{1}{2} OPT(\mathbf{v})$$

$$UTIL(\mathbf{b}) + REV(\mathbf{b}) \geq \frac{1}{2} OPT(\mathbf{v})$$

$$SW(b) \geq \frac{1}{2} OPT(\mathbf{v})$$

(λ, μ) –Smoothness via own-value deviations

Exists b_i' depending only on own value

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})$$

(λ, μ) –Smoothness via own-value deviations

Exists b_i' depending only on own value

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})$$

Note. Smoothness is property of auction not equilibrium

(λ, μ) –Smoothness via own-value deviations

Exists b_i' depending only on own value

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})$$

Applies to any auction: Not First-Price Auction specific

(λ, μ) –Smoothness implies PoA $\leq \mu/\lambda$

Proof. If $\mathbf{b}$ PNE then

$$UTIL(\mathbf{b}) + \mu \cdot REV(\mathbf{b}) \geq \boxed{\sum_i u_i(b_i', \mathbf{b}_{-\mathbf{i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})}$$

Note. $\mathrm{UTIL}(\mathbf{b}) = SW(\mathbf{b}) - REV(\mathbf{b})$

$$UTIL(\mathbf{b}) + \mu \cdot REV(\mathbf{b}) \geq \lambda \cdot OPT(\mathbf{v})$$

Note. $\mathrm{SW}(\mathbf{b}) \geq REV(\mathbf{b})$

$$SW(\mathbf{b}) + (\mu - 1) \cdot REV(\mathbf{b}) \geq \lambda \cdot OPT(\mathbf{v})$$

$$SW(\mathbf{b}) + (\mu - 1) \cdot SW(\mathbf{b}) \geq \lambda \cdot OPT(\mathbf{v})$$

$$\mu \cdot SW(\mathbf{b}) \geq \lambda \cdot OPT(\mathbf{v})$$

Finally

First Extension Theorem. If PNE PoA proved by showing (λ, μ) −smoothness property via own-value deviations, then PoA bound extends to BNE with correlated values

Note. Not specific to First-Price Auction

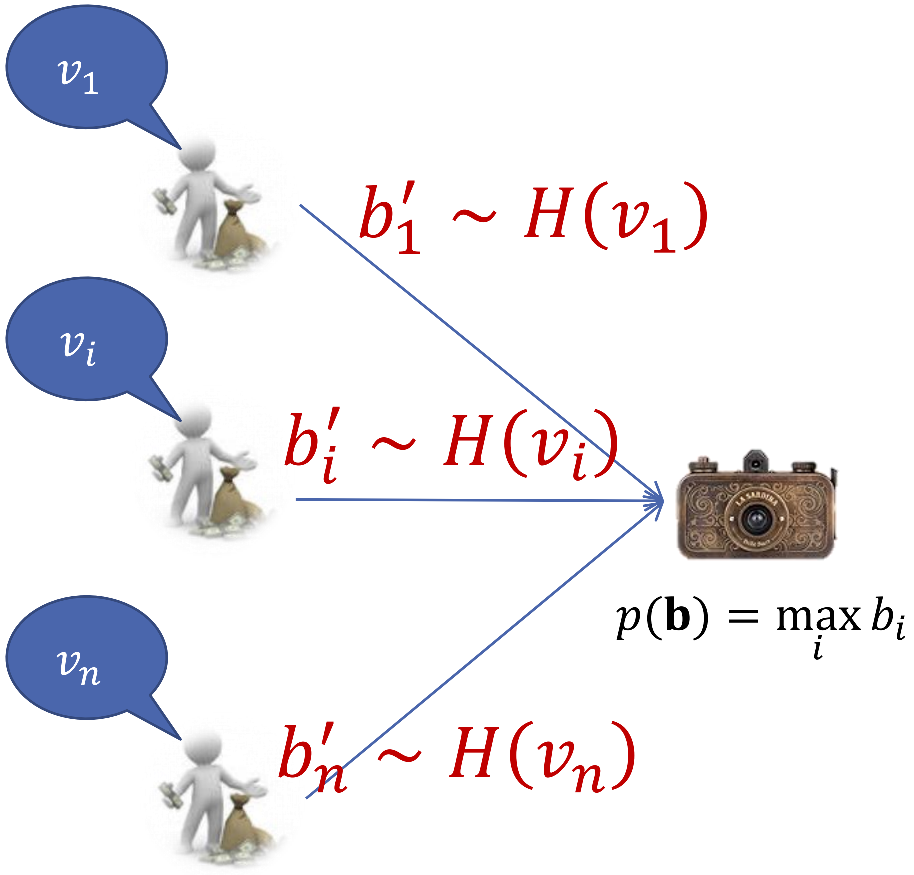
(λ, μ) –Smoothness implies BNE PoA $\leq \mu/\lambda$

Proof. If $\boldsymbol{b}(\cdot)$ BNE then $\quad E[u_i(\mathbf{b}(\mathbf{v}))] \geq E\left[u_i\left(\frac{v_i}{2}, \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right]$

$$E_v\left[UTIL(b) + \mu \cdot REV(b) \geq \boxed{\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})}\right]$$

Just redo PNE proof in expectation over values.

Optimizing over (λ, μ)

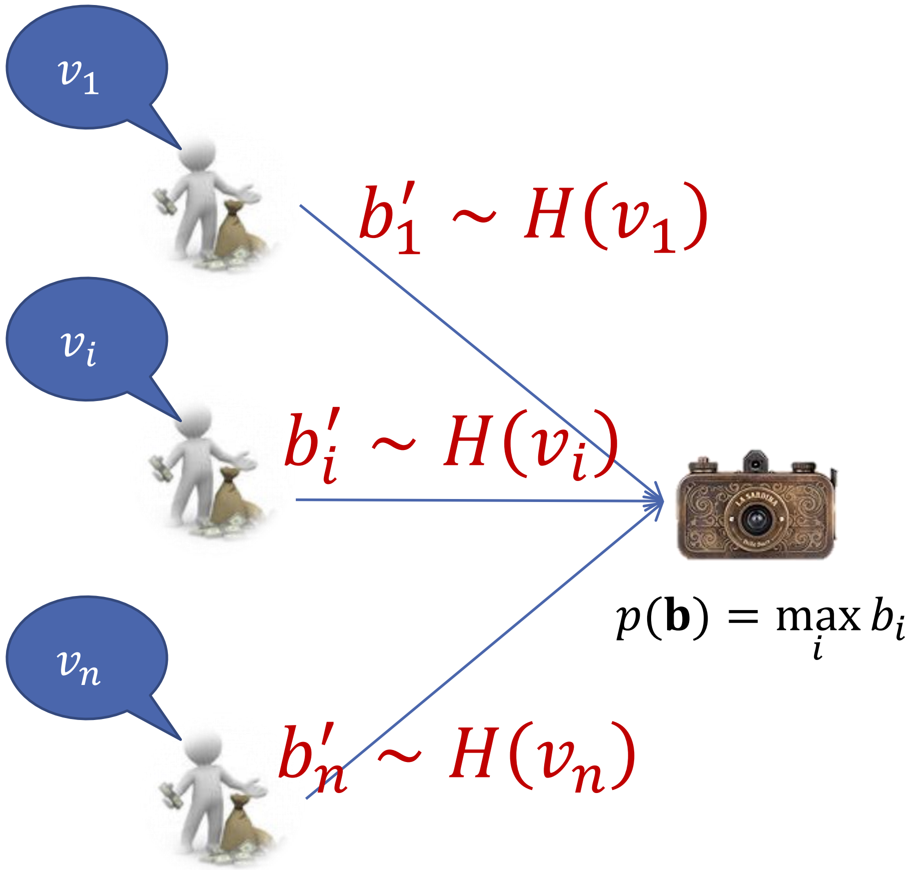


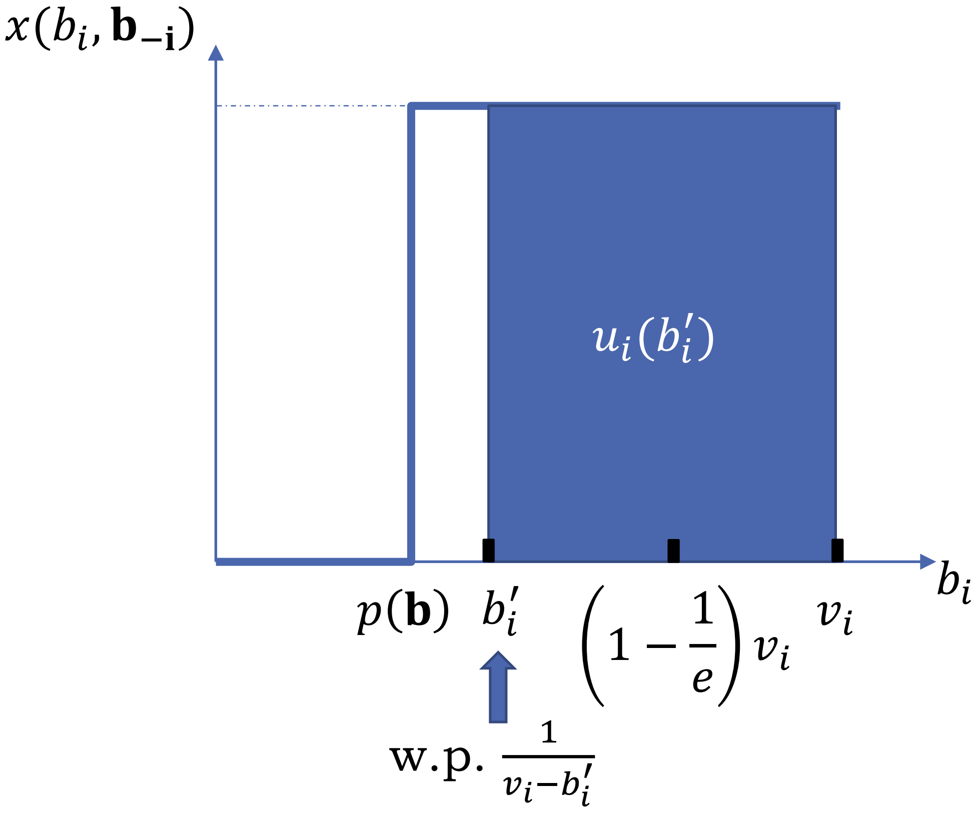
- Is half value best own-value deviation?
- Bid $b'_i \sim H(v_i)$ with support $\left[0, \left(1 - \frac{1}{e}\right) v_i\right]$ and

$$h(b'_i) = \frac{1}{v_i - b'_i}$$

Optimizing over (λ, μ)

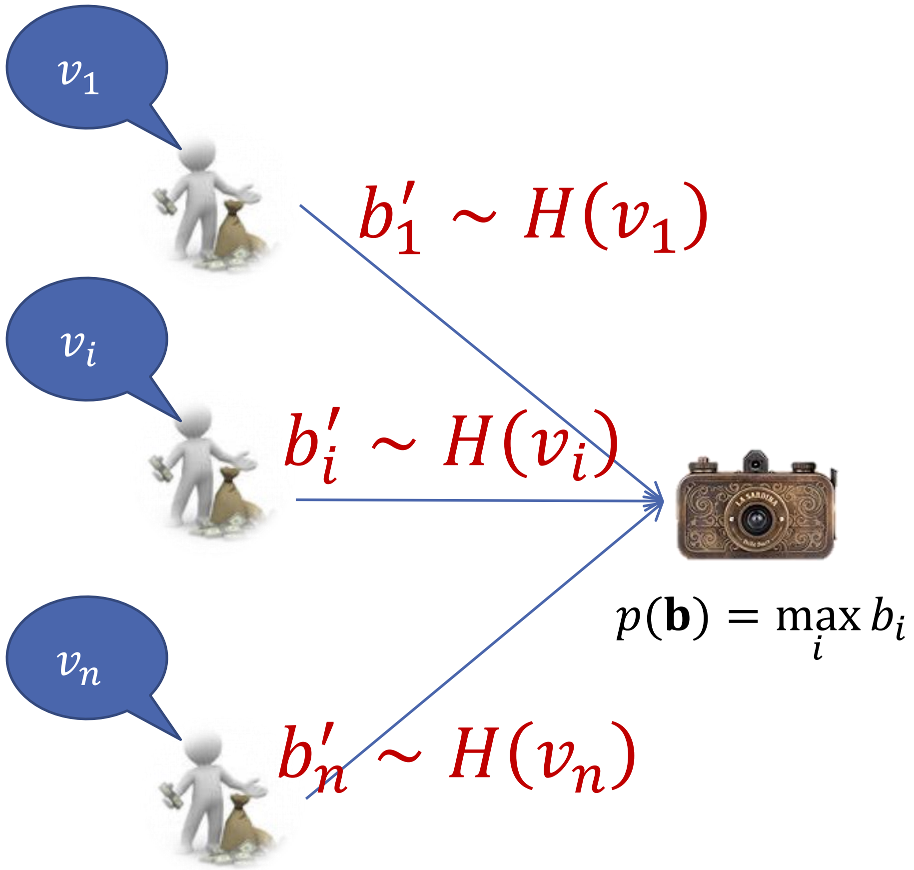
- Bid $b_i' \sim H(v_i)$ with support $\left[0, \left(1 - \frac{1}{e}\right) v_i\right]$ and $h(b_i') = \frac{1}{v_i - b_i'}$

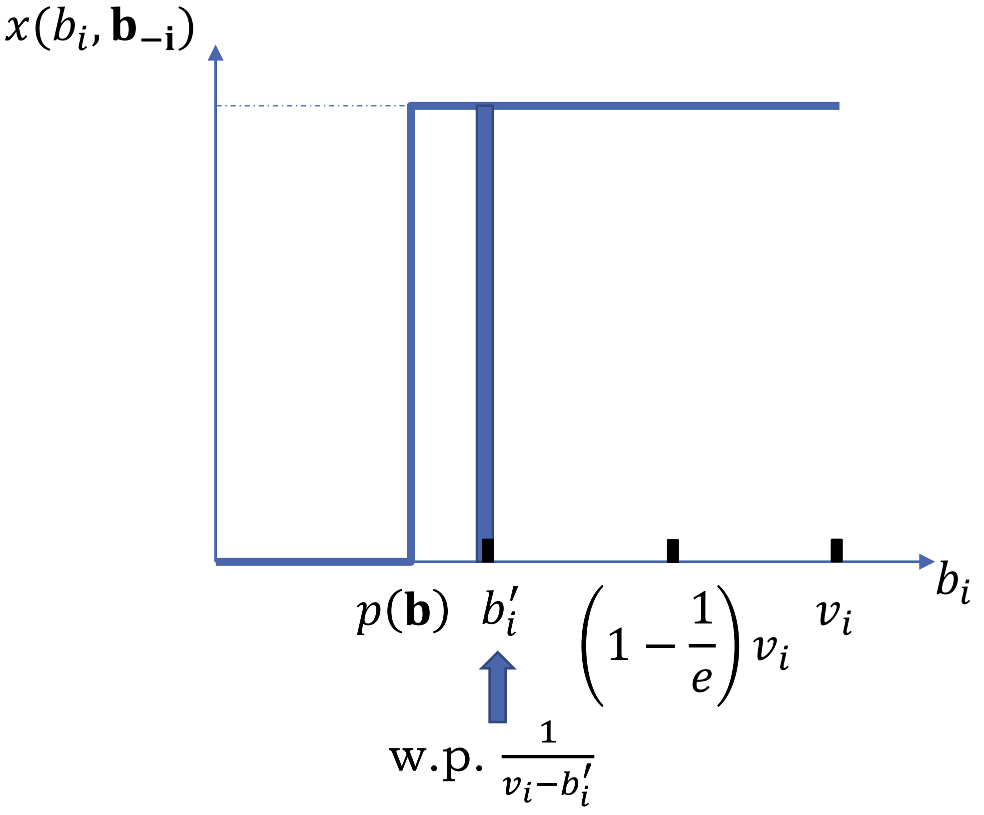




Optimizing over (λ, μ)

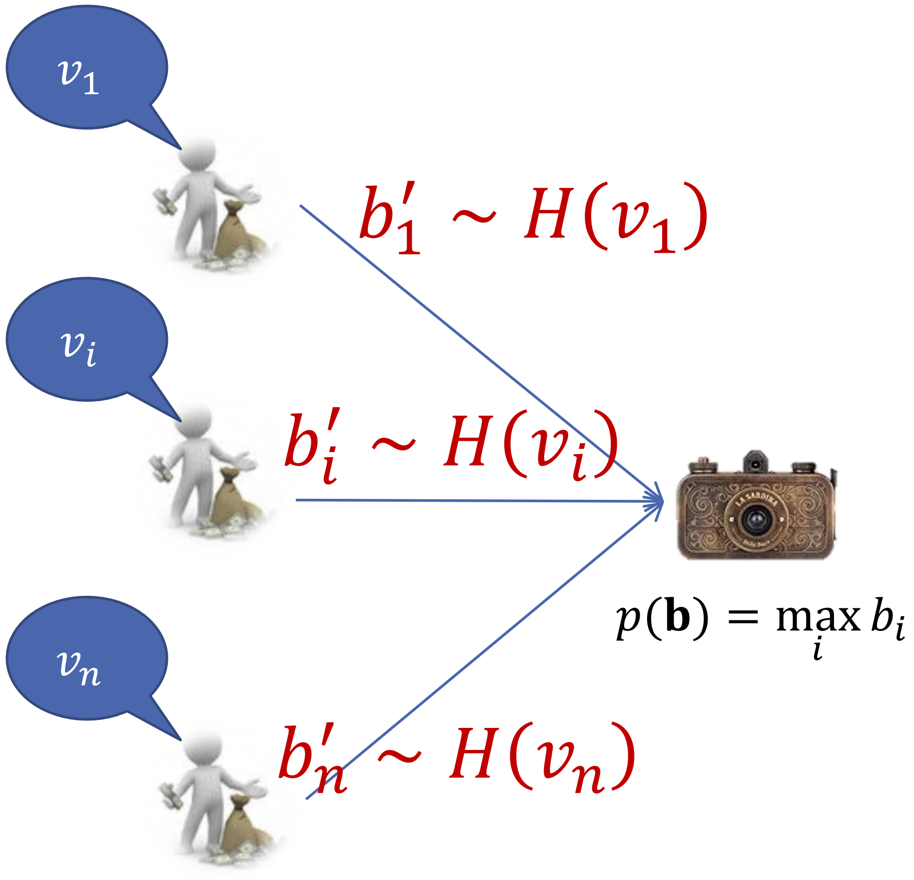
- Bid $b_i' \sim H(v_i)$ with support $\left[0, \left(1 - \frac{1}{e}\right) v_i\right]$ and $h(b_i') = \frac{1}{v_i - b_i'}$

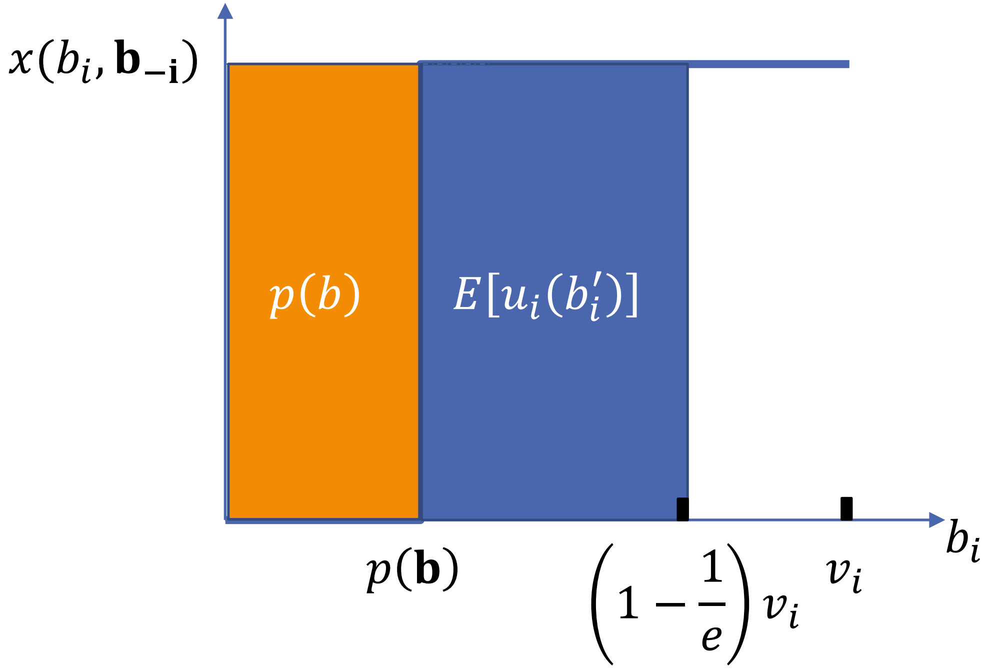




Optimizing over (λ, μ)

- Bid $b_i' \sim H(v_i)$ with support $\left[0, \left(1 - \frac{1}{e}\right) v_i\right]$ and $h(b_i') = \frac{1}{v_i - b_i'}$





$$E[u_i(b_i')] + p(b) > \left(1 - \frac{1}{e}\right) v_i$$

- So in fact: $\left(1 - \frac{1}{e}, 1\right)$-smooth. $PoA \leq \frac{e}{e-1} \approx 1.58$

RECAP

- **First Extension Thm.** If proof of PNE PoA based on (λ, μ) −smoothness via own-value based deviations then PoA of BNE with correlated values also μ/λ

QUESTIONS?

First Extension Theorem

Complete info PNE to BNE with correlated values

Second Extension Theorem

Single auction to simultaneous auctions

PNE complete information

Second Extension Theorem

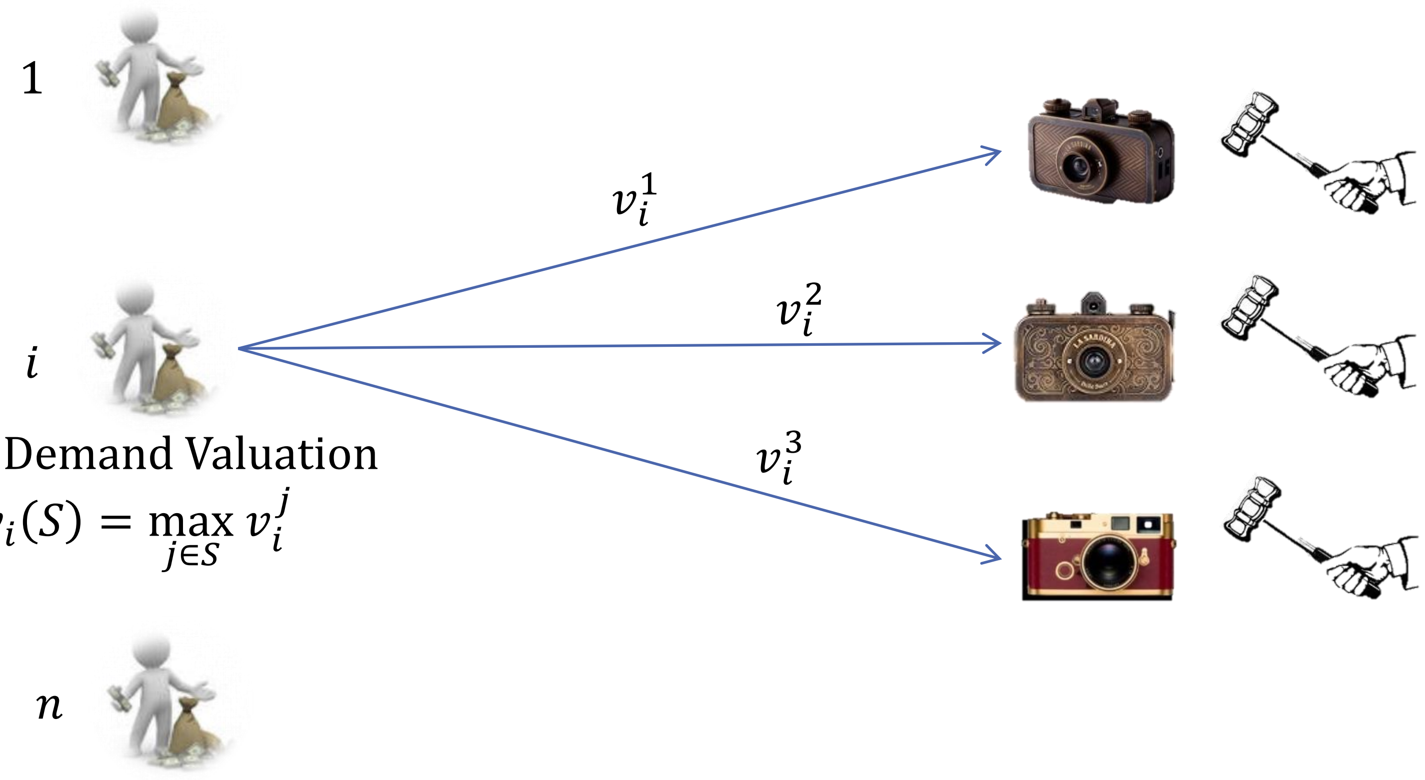
Single auction to simultaneous auctions

PNE complete information

- Target setting. Simultaneous single-item first price auctions with unit-demand bidders (complete information PNE).
- Simple setting. Single-item first price auction (complete information PNE).
- **Thm.** If proof of PNE PoA of single-item based on proving (λ, μ)-smoothness via own-value deviation then PNE PoA of simultaneous auctions also μ/λ.

References:
Roughgarden STOC'09
Roughgarden EC'12
Syrgkanis '12
Syrgkanis, Tardos STOC'13

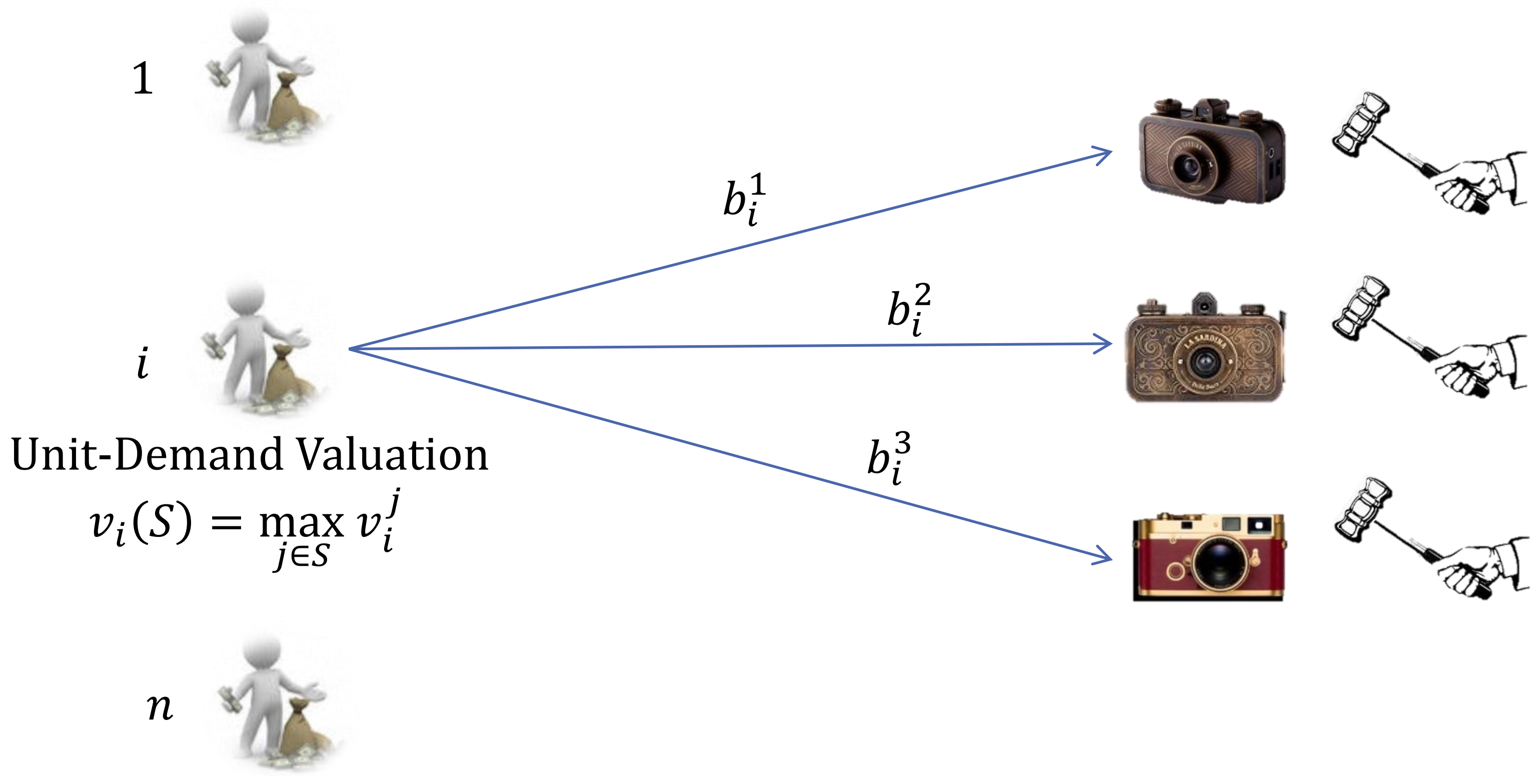
Simultaneous First-Price Auctions
Unit-demand bidders



Unit-Demand Valuation

$$v_i(S) = \max_{j \in S} v_i^j$$

Simultaneous First-Price Auctions
Unit-demand bidders



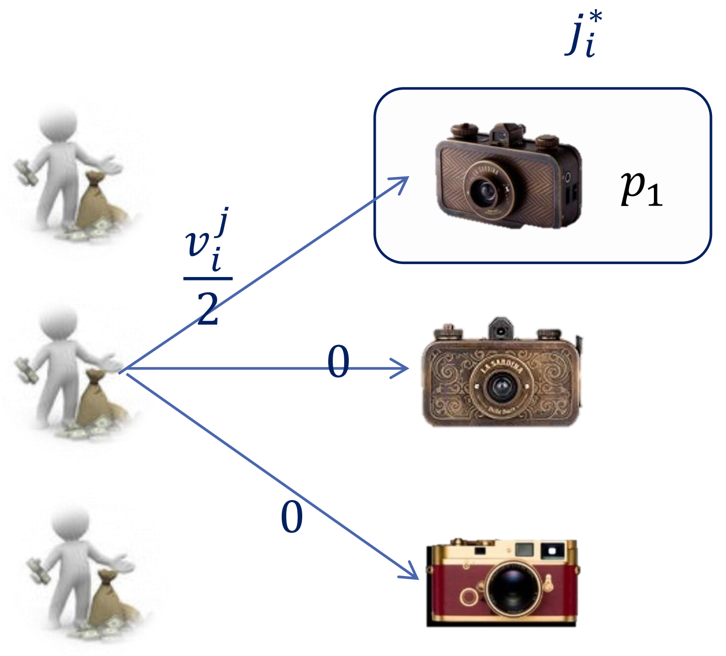
Simultaneous First-Price Auctions

Can we derive global efficiency guarantees from local $\left(\frac{1}{2}, 1\right)$ –smoothness of each first price auction?

APPROACH: Prove smoothness of the global mechanism

GOAL: Construct global deviation

IDEA: Pick your item in the optimal allocation and perform the smoothness deviation for your local value v_i^j, i.e. $\frac{v_i^j}{2}$



Simultaneous First-Price Auctions

Smoothness locally:

$$u_i(b_i', \mathbf{b_{-i}}) + p_{j_i^*}(\mathbf{b}) \geq \frac{v_i^{j_i^*}}{2}$$

Summing over players:

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + REV(\mathbf{b}) \geq \frac{1}{2} \cdot OPT(\mathbf{v})$$

Implying $\left(\frac{1}{2}, 1\right)-$smoothness property globally.

Second Extension Theorem. If proof of PNE PoA of single-item auction based on proving (λ, μ)-smoothness smoothness via own-value deviation then PNE PoA of simultaneous auctions also $\leq \mu/\lambda$.

BNE PoA?

- BNE PoA of simultaneous single-item auctions with correlated unit-demand values $\leq 1/2$?
- Not really: deviation not oblivious to opponent valuations
- Item in the optimal matching depends on values of opponents

But Half-way there

- What we showed:

Exists b_i' depending only on valuation profile $\mathbf{v}$ (not $\mathbf{b_{-i}}$)

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \mu \cdot REV(\mathbf{b}) \geq \lambda \cdot OPT(\mathbf{v})$$

RECAP

Second Extension Theorem. If proof of PNE PoA of single-item auction based on proving (λ, μ)-smoothness then PNE PoA of simultaneous auctions also $\leq \mu/\lambda$.

Next we will extend above to BNE

QUESTIONS?

Second Extension Theorem

Single auction to simultaneous auctions

PNE complete information

43

Third Extension Theorem

Complete info PNE to BNE with independent values

- Target setting. First Price Bayes-Nash Equilibrium with asymmetric **independent** values

- Simple setting. Complete information Pure Nash Equilibrium

- **Thm.** If proof of PNE PoA based on (λ, μ)-smoothness via valuation profile dependent deviation then PoA of BNE with independent values also μ/λ

Third Extension Theorem

Complete info PNE to BNE with independent values

References:

Christodoulou et al. ICALP'08
Roughgarden EC'12
Syrgkanis '12
Syrgkanis, Tardos STOC'13

Does this extend to BNE PoA?

(λ, μ) −Smoothness via valuation profile deviations

Exists b_i' depending only on valuation profile $\mathbf{v}$ (not $\mathbf{b_{-i}}$)

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})$$

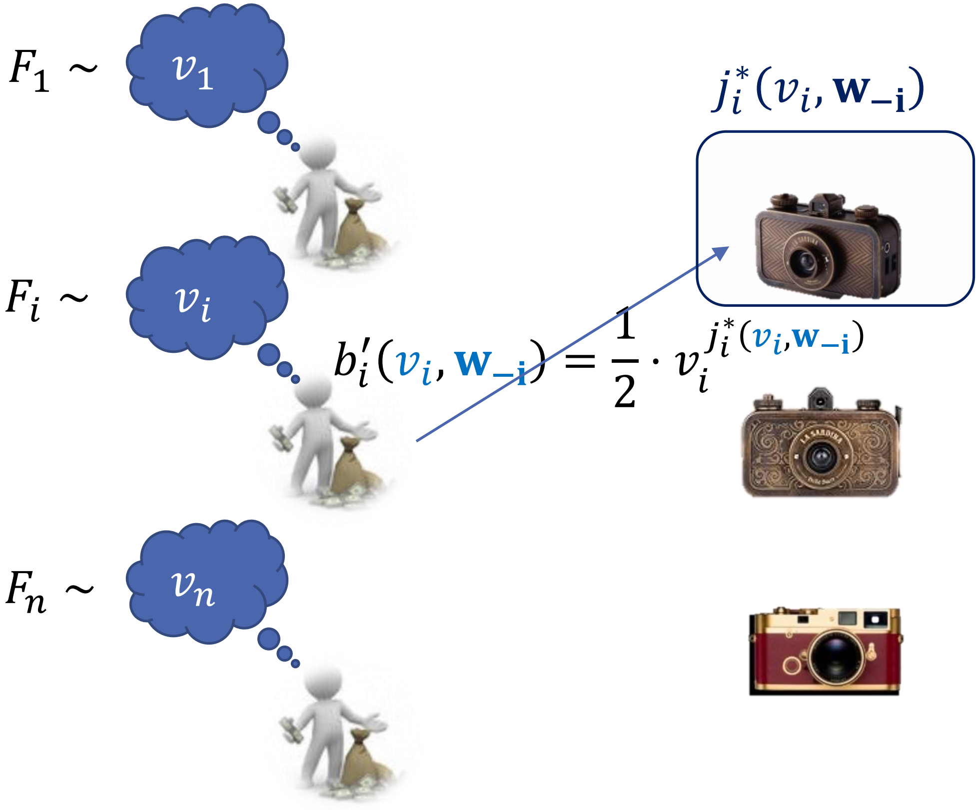
Recall First Extension Theorem.

If PNE PoA proved by showing (λ, μ) −smoothness property via own-value deviations, then PoA bound extends to BNE with correlated values

- Relax First Extension Theorem to allow for dependence on opponents values

- To counterbalance: assume independent values

BNE (independent valuations)



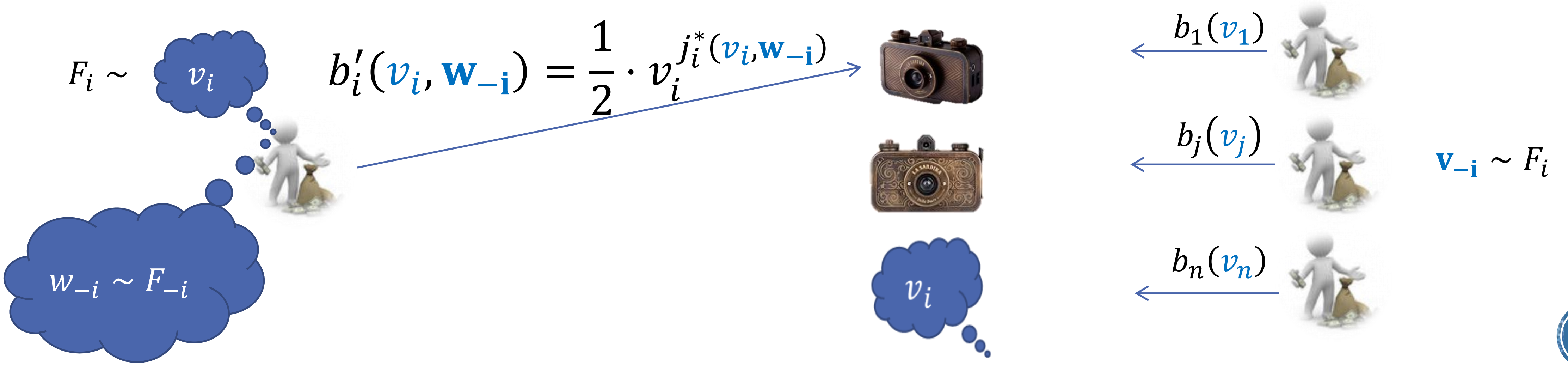
- Need to construct feasible BNE deviations
- Each player random samples the others values and deviates as if that was the true values of his opponents
- Above works out, due to independence of value distributions

BNE (independent valuations)

$$\underbrace{E\left[u_i^{v_i}\left(b_i'(v_i, \mathbf{w_{-i}}), \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right]} = \underbrace{E\left[u_i^{w_i}\left(b_i'(\mathbf{w}), \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right]}$$

Utility of deviation of player i
In expectation over his own
value too.

Utility of deviation from a random sample of player i who knows the values of all other players.
But where players play non equilibrium strategies.



BNE (independent valuations)

$$E\left[u_i^{v_i}\left(b_i'(v_i, \mathbf{w}_{-\mathbf{i}}), \mathbf{b}_{-\mathbf{i}}(\mathbf{v}_{-\mathbf{i}})\right)\right] = E\left[u_i^{w_i}\left(b_i'(\mathbf{w}), \mathbf{b}_{-\mathbf{i}}(\mathbf{v}_{-\mathbf{i}})\right)\right]$$

Utility of deviation of player i
In expectation over his own
value too.

Utility of deviation from a random sample of player i who knows the values of all other players.
But where players play non equilibrium strategies.

$F_i \sim$

$$b_i'(\mathbf{w}) = \frac{1}{2} \cdot v_i^{j_i^*(\mathbf{w})}$$

$b_1(v_1)$

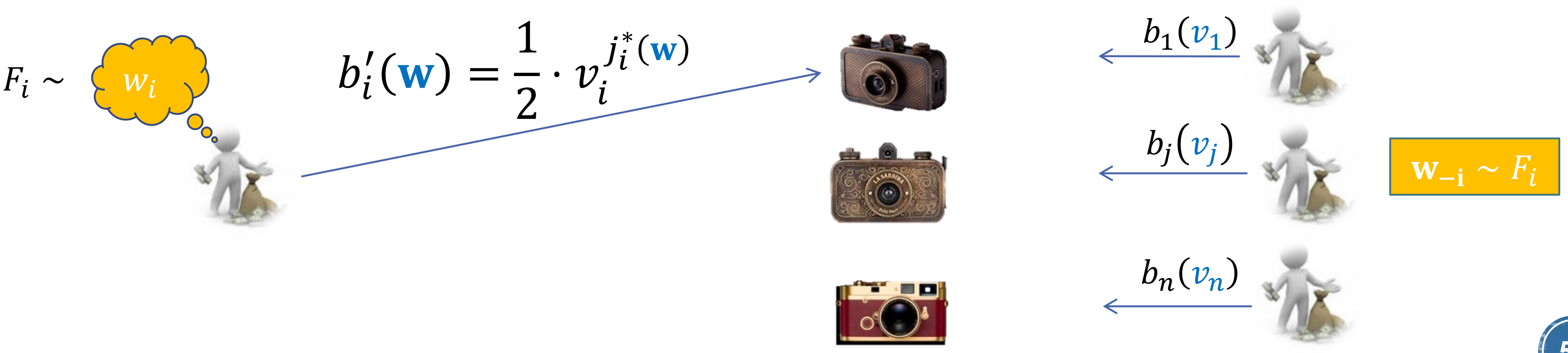
$b_j(v_j)$

$b_n(v_n)$

$\mathbf{w}_{-\mathbf{i}} \sim F_i$

BNE (independent valuations)

$$\underbrace{\sum_i E\left[u_i^{v_i}\left(b_i'(v_i,\mathbf{w}_{-\mathbf{i}}),\mathbf{b}_{-\mathbf{i}}(\mathbf{v}_{-\mathbf{i}})\right)\right]}_{\text{Sum of deviating utilities}} = E\left[\underbrace{\sum_i u_i^{w_i}\left(b_i'(\mathbf{w}),\mathbf{b}_{-\mathbf{i}}(\mathbf{v}_{-\mathbf{i}})\right)}_{\text{Sum of complete information setting deviating utilities}}\right]$$



…lent valuations)

Recall. Exists b_i' depending only on valuation profile $\mathbf{v}$ (not $\mathbf{b_{-i}}$)

$$\ldots(\mathbf{v_{-i}}))] = E\left[\sum_i u_i^{w_i}\left(b_i'(\mathbf{w}), \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right]$$

For any bid vector $\mathbf{b}$

Utility of deviation of player i $\quad \geq E\left[\underbrace{\lambda \cdot OPT(\mathbf{w}) - \mu \cdot REV\left(\mathbf{b}(\mathbf{v})\right)}_{\text{By smoothness on the left}}\right]$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})$$

$F_i \sim$ w_i

$$b_i'(\mathbf{w}) = \frac{1}{2} \cdot v_i^{j_i^*(\mathbf{w})}$$

$b_1(v_1)$

$b_j(v_j)$

$b_n(v_n)$

$\mathbf{w_{-i}} \sim F_i$

$$u_i\left(b_i'(\mathbf{w}), \mathbf{b_{-i}}(\mathbf{v_{-i}})\right) \geq \frac{1}{2} \cdot v_i^{j_i^*(\mathbf{w})} - p_{j_i^*(\mathbf{w})}\left(\mathbf{b}(\mathbf{v})\right)$$

BNE (independent valuations)

$$\sum_i E\left[u_i^{v_i}\left(b_i'(v_i, \mathbf{w_{-i}}), \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right] = E\left[\sum_i u_i^{w_i}\left(b_i'(\mathbf{w}), \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right]$$

$$\geq E\left[\lambda \cdot OPT(\mathbf{w}) - \mu \cdot REV\left(\mathbf{b}(\mathbf{v})\right)\right]$$

Found b_i' that depend only on v_i such that:

$$\sum_i E\left[u_i\left(b_i'(v_i), \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right] + \mu \cdot E\left[REV\left(\mathbf{b}(\mathbf{v})\right)\right] \geq \lambda \cdot E[OPT(\mathbf{v})]$$

Rest is easy

Third Extension Theorem. If PNE PoA proved by showing (λ, μ) −smoothness property via valuation profile dependent deviations, then PoA bound extends to BNE with independent values

RECAP

- **Thm.** If proof of PNE PoA based on (λ, μ)-smoothness via valuation profile dependent deviation then PoA of BNE with independent values also μ/λ

- **Corollary.** If PNE PoA of single-item auction proved via (λ, μ)-smoothness via valuation profile dependent deviation, then BNE of simultaneous auctions with unit-demand and independent also μ/λ

Third Extension Theorem

Complete info PNE to BNE with independent values

RECAP

- **Thm.** If proof of PNE PoA based on (λ, μ)-smoothness via valuation profile dependent deviation then PoA of BNE with independent values also μ/λ
- **Corollary.** If PNE PoA of single-item auction proved via (λ, μ)-smoothness via valuation profile dependent deviation, then BNE of simultaneous auctions with **submodular** and independent also μ/λ
- **Corollary.** BNE PoA of simultaneous first price auctions with submodular bidders $\leq \frac{e}{e-1}$

QUESTIONS?

Third Extension Theorem

Complete info PNE to BNE with independent values

56

Direct approach: Arguing about distributions

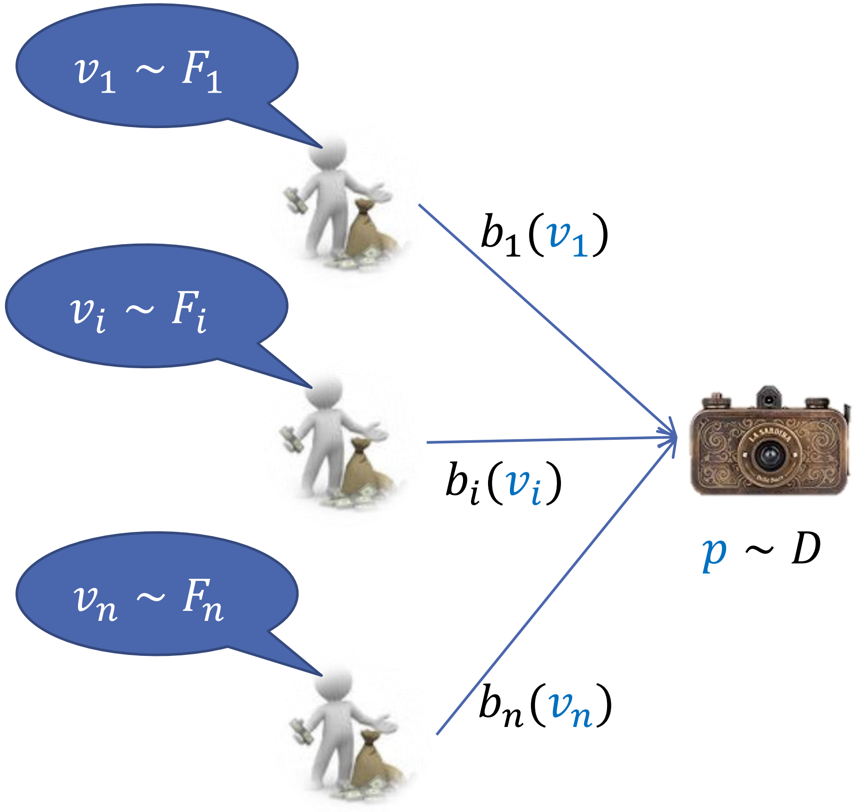
Direct approach

Arguing about distributions

- Focusing on complete info PNE, might be restrictive in some settings
- Working with the distributions directly can potentially yield better bounds

References:
Feldman et al. STOC’13

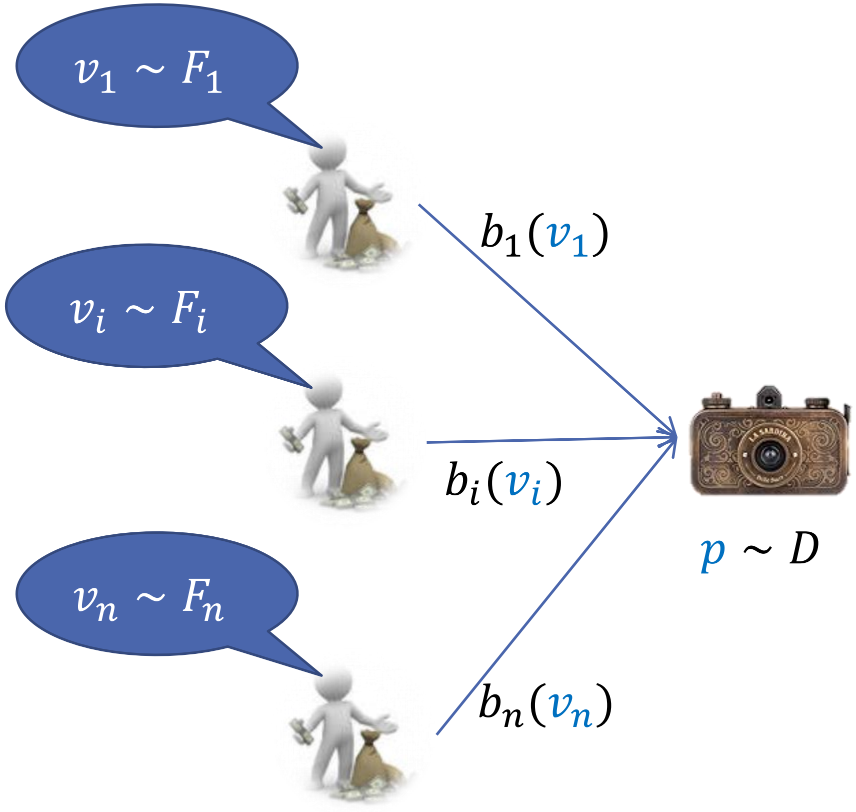
Single-item auction BNE



- Price of the item follows a distribution D
- What if a player deviates to bidding a random sample from price distribution
- The probability that he wins is ½ by symmetry of the two distributions
- He pays at most $E[p]$

$$E\left[u_i\left(b_i', \mathbf{b_{-i}}(\mathbf{v_{-i}})\right)\right] \geq \frac{v_i}{2} - E[p]$$

Single-item auction BNE

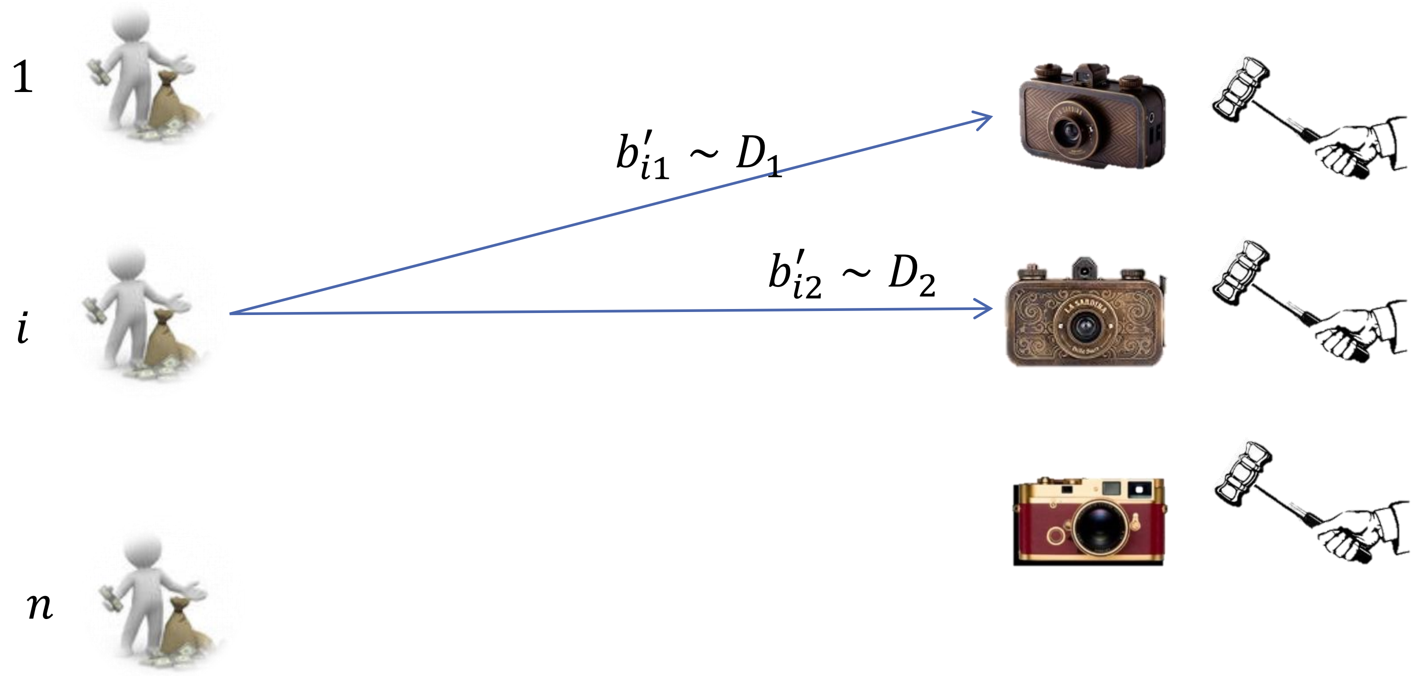


- Same spirit: exists deviations that depend on price distribution such that
$$\sum_i E\left[u_i\left(b_i', \mathbf{b}_{-\mathbf{i}}(\mathbf{v}_{-\mathbf{i}})\right)\right] + E\left[REV\left(\mathbf{b}(\mathbf{v})\right)\right] \geq \frac{E[OPT(\mathbf{v})]}{2}$$

- BNE PoA≤ 2

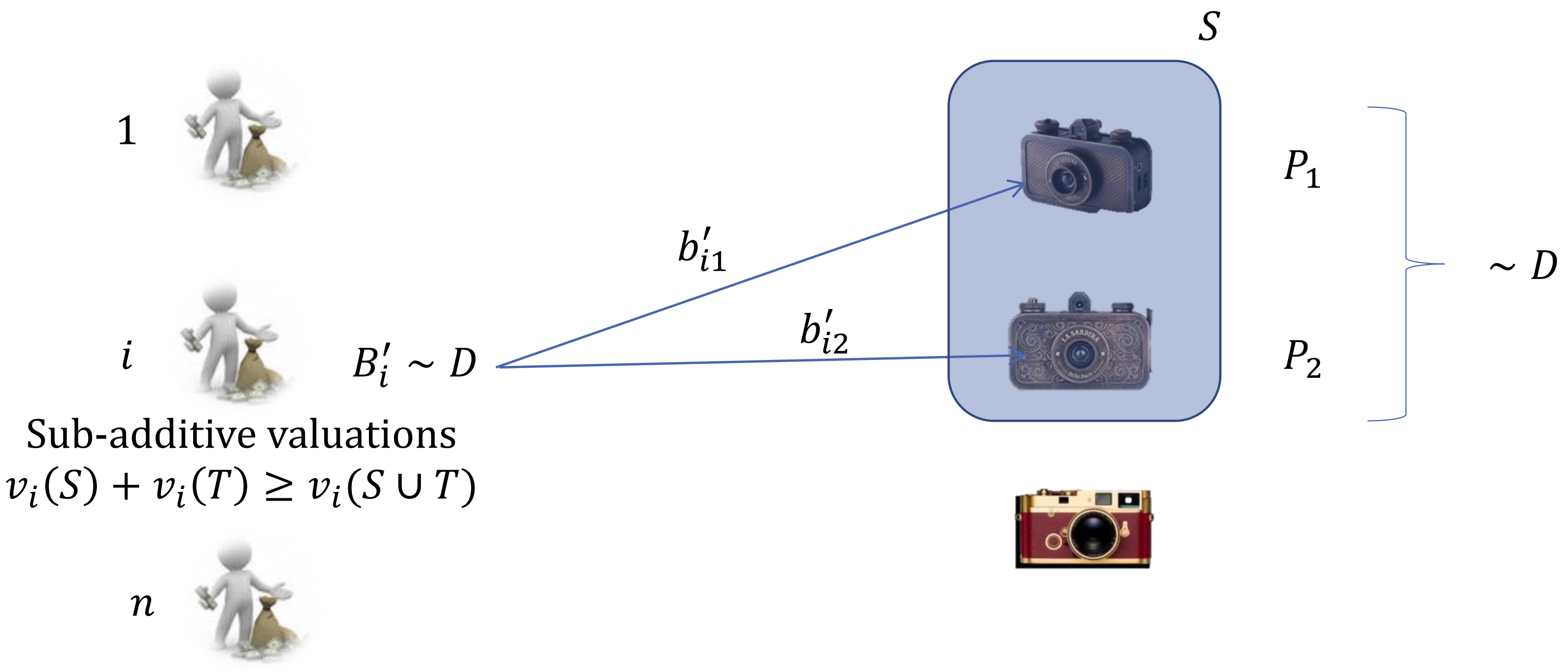
What does it buy us

- Correlated deviating strategies across multiple auctions
- Decomposition of deviation analysis to separate deviations imposes independent randomness



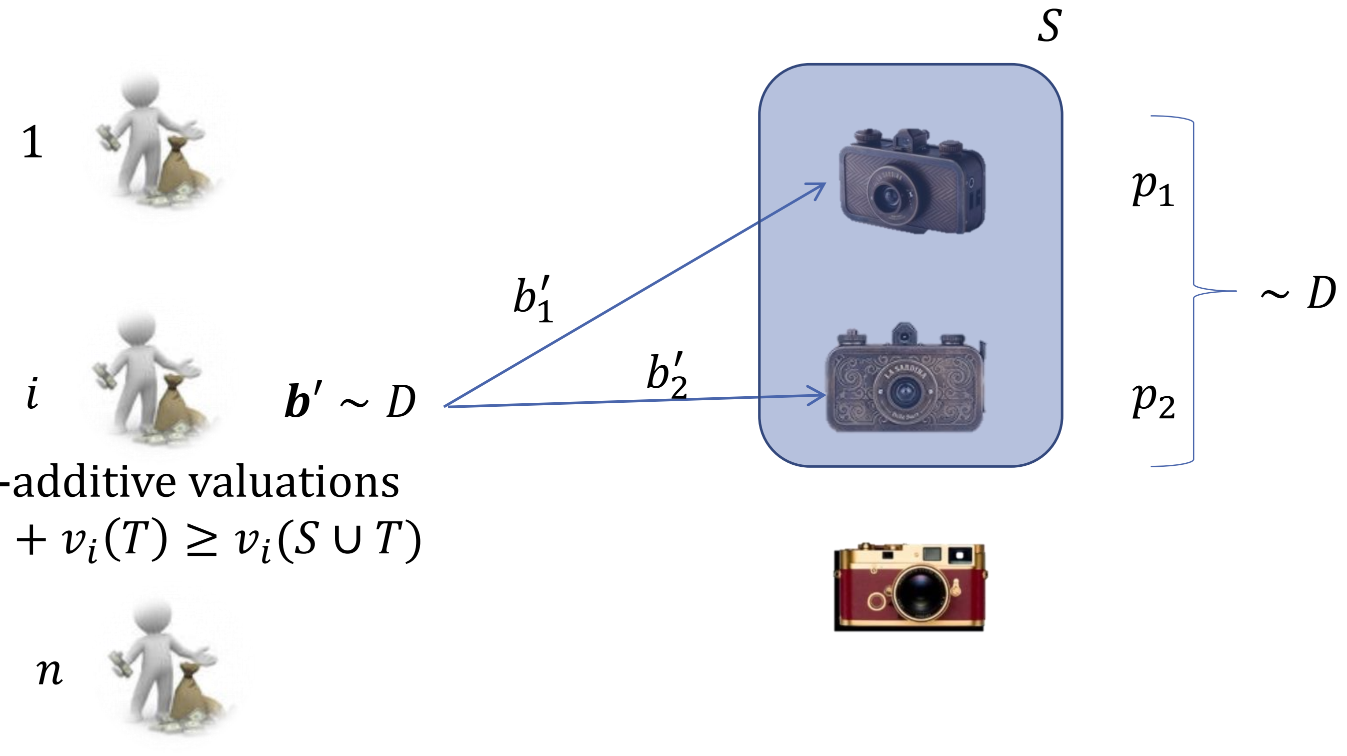
What does it buy us

- Correlation can achieve higher deviating utility



What does it buy us

- Correlation can achieve higher deviating utility



Sub-additive valuations
$v_i(S) + v_i(T) \geq v_i(S \cup T)$

- Draw bid from price distribution
- $X(b, p)$: set of won items with bid vector b and price vector p
- Either I win or price wins:
  $$X(b,p) + X(p,b) = S$$
- By symmetry:
  $$E\big[v\big(X(b',p)\big)\big] = E\big[v\big(X(p,b')\big)\big]$$

- Value collected: $E\big[v\big(X(b',p)\big)\big] = \frac{1}{2}E\big[v\big(X(b',p)\big) + v\big(X(p,b')\big)\big] \geq \frac{1}{2}E[v(S)]$

Direct approach

Arguing about distributions

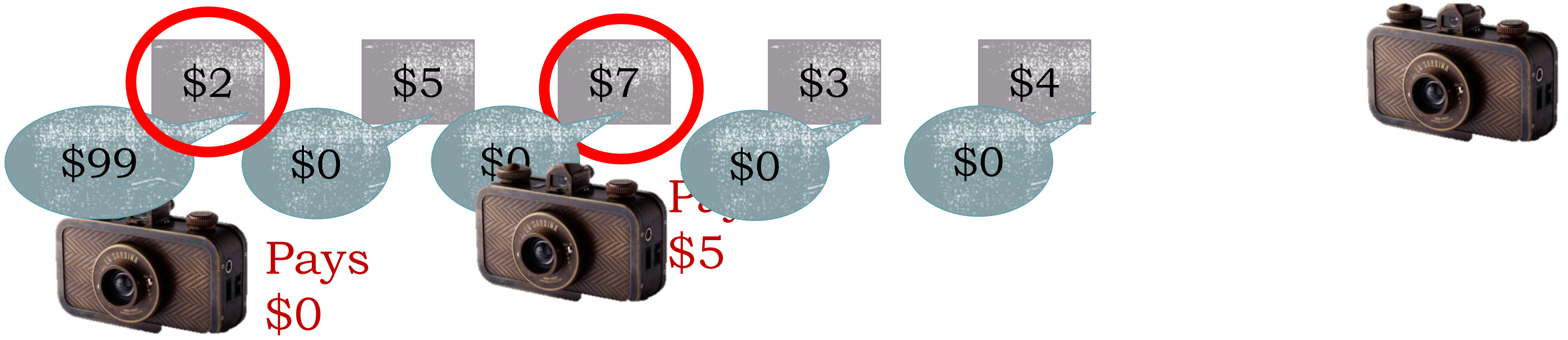
- Drawing deviation from price distribution!
- Buys correlation across auctions
- Better bounds beyond submodular

64

Second Price Payment Rules

Second price

Vickrey Auction - Truthful, efficient, simple (second price)



but has many bad Nash equilibria

Assume bid ≤ value (no overbidding)

Theorem. All Nash equilibria efficient. highest value wins

Second Price and Overbidding

- Same approach but replace Payments with "Winning Bids" and use no-overbidding

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot BIDS(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot OPT(\mathbf{v})$$

- No overbidding assumption:

$$BIDS \leq WELFARE$$

Then $PoA \leq \frac{1+\mu}{\lambda}$

Smoothness of Vickrey Auction

- Deviate to bidding your value: $b_i'(v_i) = v_i$
- $B(\mathbf{b})$: winning bid
- Either winning bid $\mathrm{B}(\mathbf{b}) \geq v_i$ or $u_i(b_i', \mathbf{b_{-i}}) = v_i - B_i(\mathbf{b})$

$$u_i(b_i', \mathbf{b_{-i}}) + B_i(\mathbf{b}) \geq v_i \Rightarrow u_i(b_i', \mathbf{b_{-i}}) + B_i(\mathbf{b}) \cdot x_i^*(\mathbf{v}) \geq v_i \cdot x_i^*(\mathbf{v})$$

$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + BIDS(\mathbf{b}) \geq OPT(\mathbf{v})$$

Smoothness of Vickrey Auction

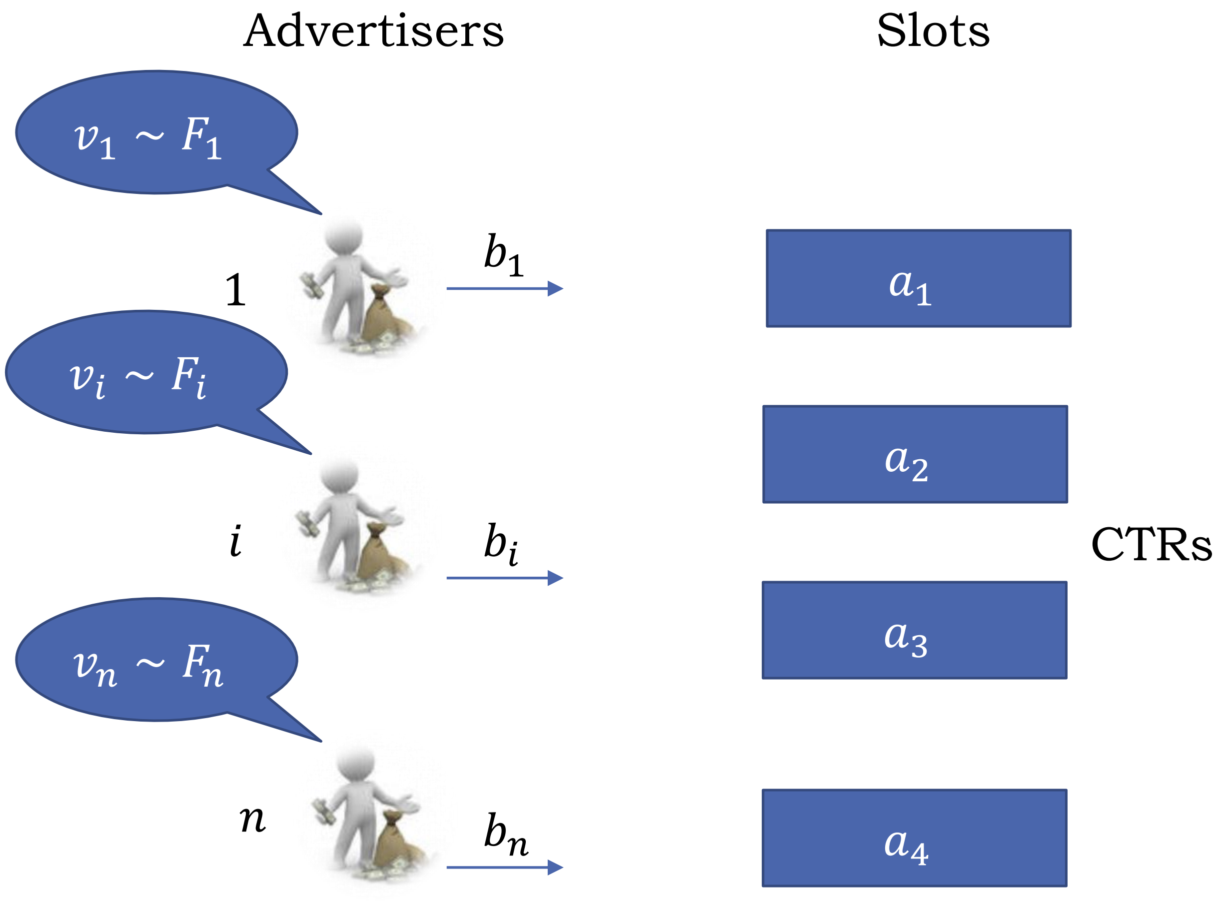
- Vickrey auction (1,1)-smooth using bids
- $PoA \leq 2$: under no-overbidding

- Vickrey is efficient?

- $PoA \leq 2$: extends to simultaneous Vickrey auctions even under BNE with independent values

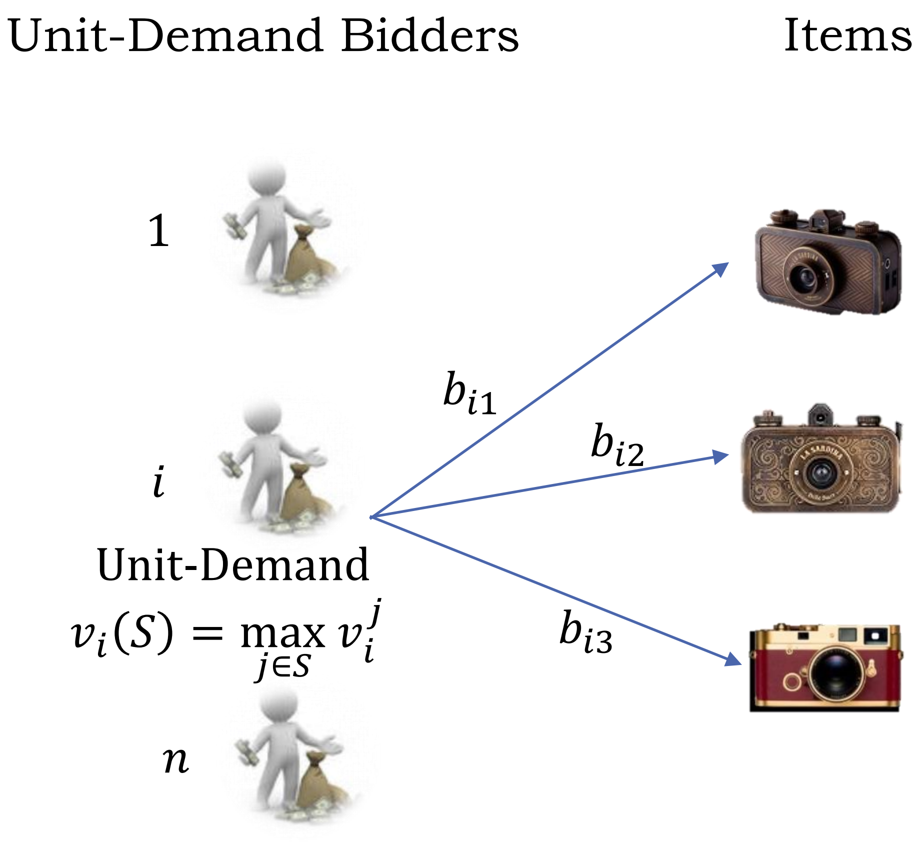
Sneak Peek of Examples

Generalized First-Price Auction



- Allocate slots by bid
- Charge bid per-click
- Utility:
$$u_i(b) = a_{\sigma(i)}(v_i - b_i)$$

Matching Markets – Greedy Mechanism



- Allocated items greedily to highest remaining bid
- If allocated item $j(b)$, charge $b_i^{j(b)}$
- Utility:

$$u_i(b) = v_i^{j(b)} - b_i^{j(b)}$$

Single-Minded Combinatorial Auction

Single-Minded Bidders

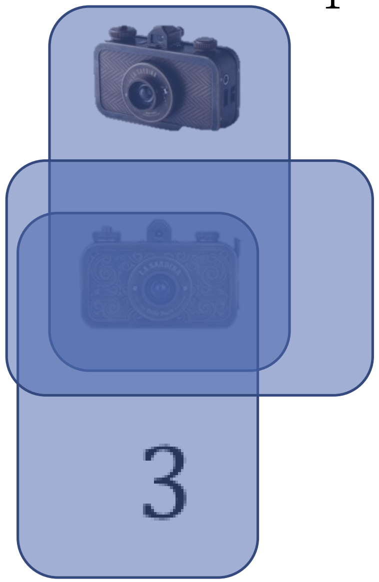
Items

1 

2 

Single-minded:
v_i for whole set S_i

3 



- Each bidder submits b_i and T_i
- Run some algorithm (optimal or greedy $O(\sqrt{m})$-approx.) over reported single-minded values
- Charge bid b_i if allocated

Examples

GFP

- Allocate slots by bid
- Charge bid per-click
- Utility: $u_i(b) = a_{\sigma(i)}(v_i - b_i)$

Matching Markets-Greedy Allocation

- Allocated items greedily to highest remaining bid
- If allocated item $j(b)$, charge $b_i^{j(b)}$
- Utility: $u_i(b) = v_i^{j(b)} - b_i^{j(b)}$

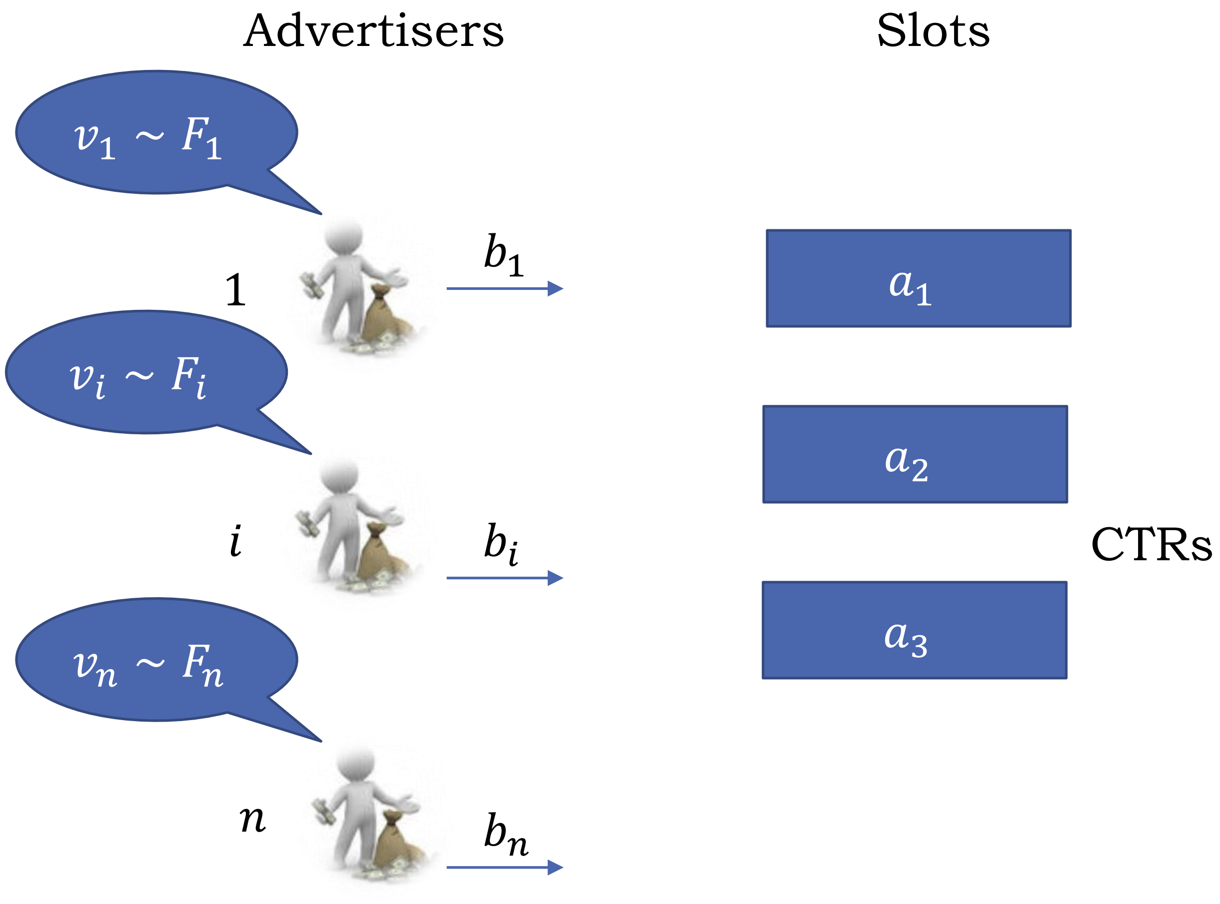
Single-Minded Combinatorial Auctions

- Each bidder submits b_i and T_i
- Run some algorithm (optimal or greedy $O(\sqrt{m})$-approx.) over reported single-minded values
- Charge bid b_i if allocated

74

Examples

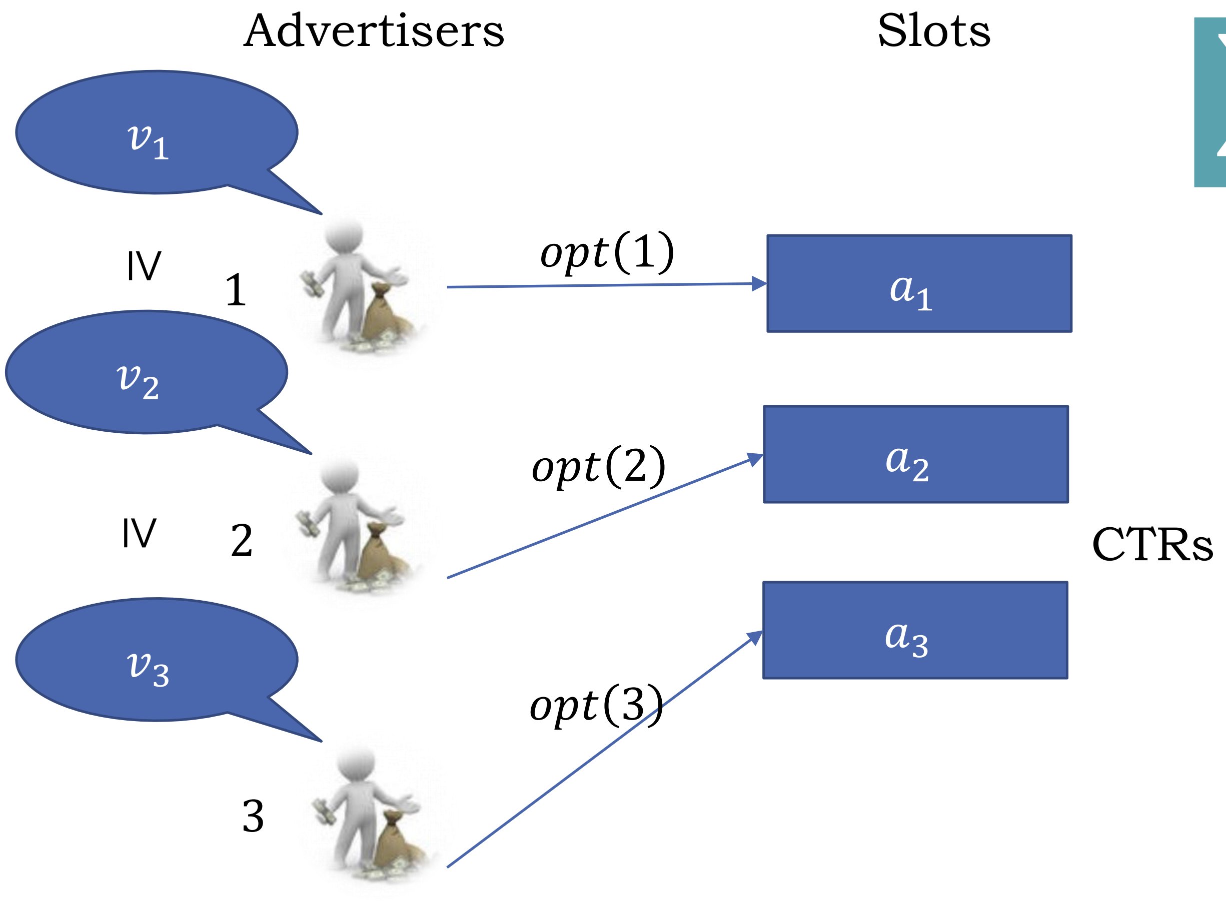
Generalized First-Price Auction



- Allocate slots by bid
- Charge bid per-click
- Utility:

$$u_i(b) = a_{\sigma(i)}(v_i - b_i)$$

Smoothness of GFP



$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \mu \cdot REV(\mathbf{b}) \geq \lambda \cdot \sum_i a_{opt(i)} v_i$$

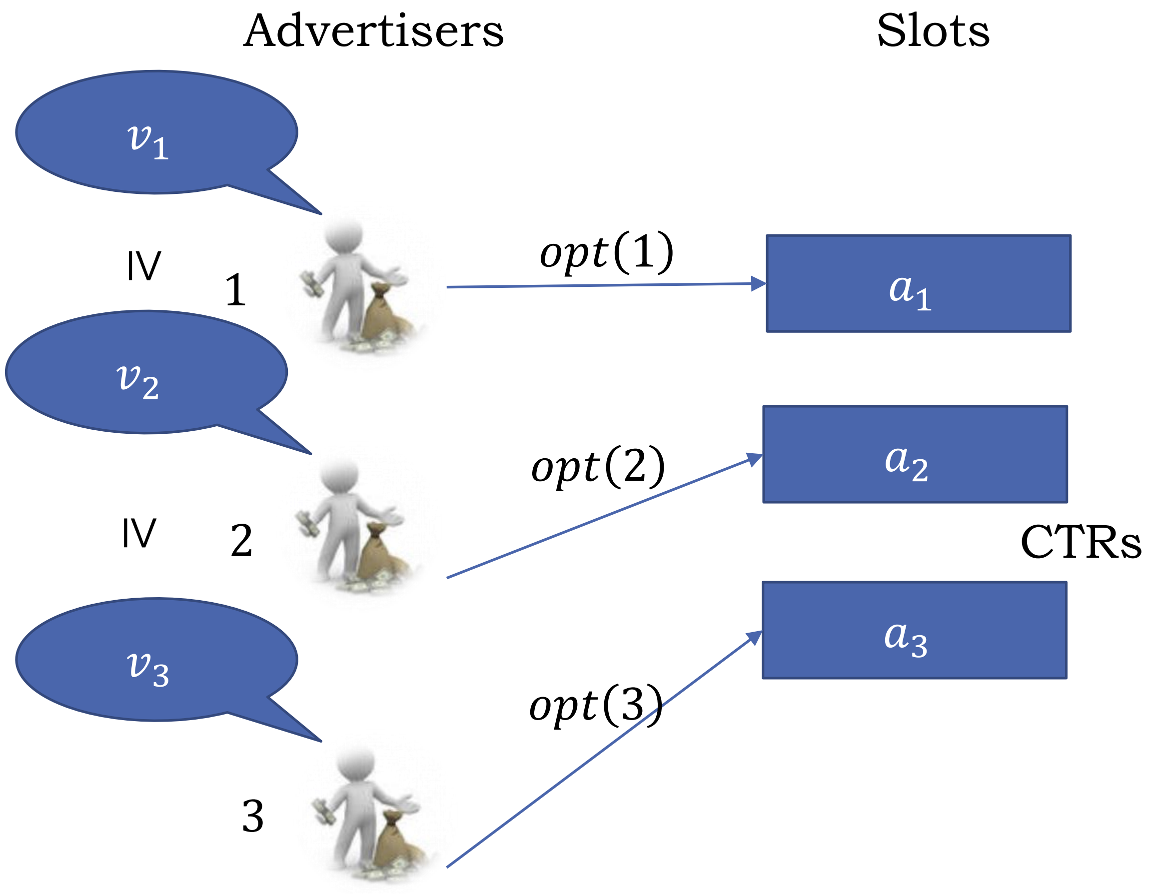
- $b_i' = \frac{v_i}{2}$
- Either bid of player at slot opt$(i) \geq \frac{v_i}{2}$
- Or utility $\geq \frac{a_{opt(i)} v_i}{2}$

$$u_i\left(\frac{v_i}{2}, b_{-i}\right) + a_{opt(i)} \cdot b_{\pi(opt(i))} \geq \frac{a_{opt(i)} v_i}{2}$$

$$\sum_i u_i\left(\frac{v_i}{2}, b_{-i}\right) + \sum_i a_{opt(i)} \cdot b_{\pi(opt(i))} \geq \sum_i \frac{a_{opt(i)} v_i}{2}$$

$$\sum_i u_i\left(\frac{v_i}{2}, b_{-i}\right) + REV(b) \geq \frac{1}{2} \cdot OPT(v)$$

Smoothness of GFP



$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot \sum_i a_{opt(i)} v_i$$

$$\sum_i u_i\left(\frac{v_i}{2}, b_{-i}\right) + REV(b) \geq \frac{1}{2} \cdot OPT(v)$$

Thm. $PoA \leq 2$

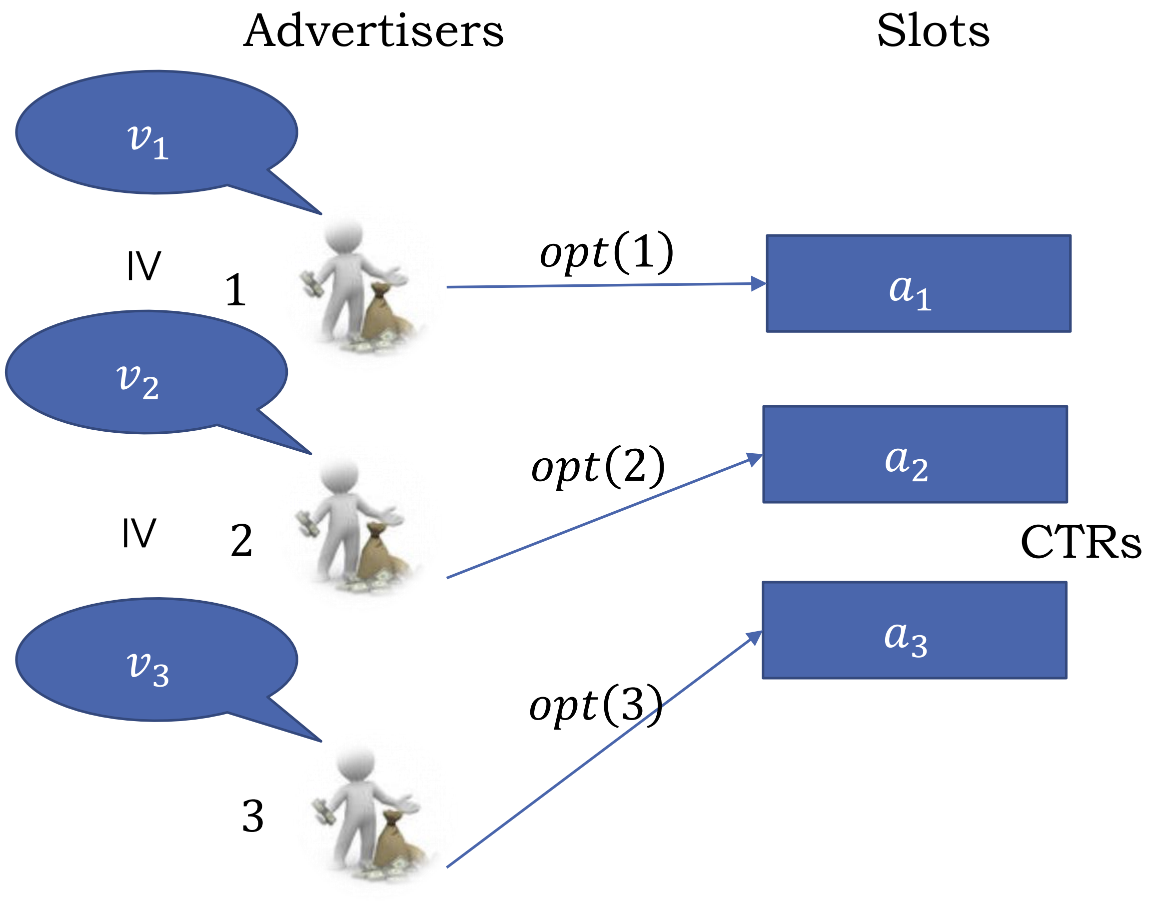
Proof.

$$\sum_i u_i(b) \geq \sum_i u_i\left(\frac{v_i}{2}, b_{-i}\right)$$

$$UTIL(b) + REV(b) \geq \frac{1}{2} \cdot OPT(v)$$

$$SW(v) \geq \frac{1}{2} \cdot OPT(v)$$

Smoothness of GFP



$$\sum_i u_i(b_i', \mathbf{b_{-i}}) + \boldsymbol{\mu} \cdot REV(\mathbf{b}) \geq \boldsymbol{\lambda} \cdot \sum_i a_{opt(i)} v_i$$

$$\sum_i u_i\left(\frac{v_i}{2}, b_{-i}\right) + REV(b) \geq \frac{1}{2} \cdot OPT(v)$$

Thm. Bayes-Nash $PoA \leq 2$

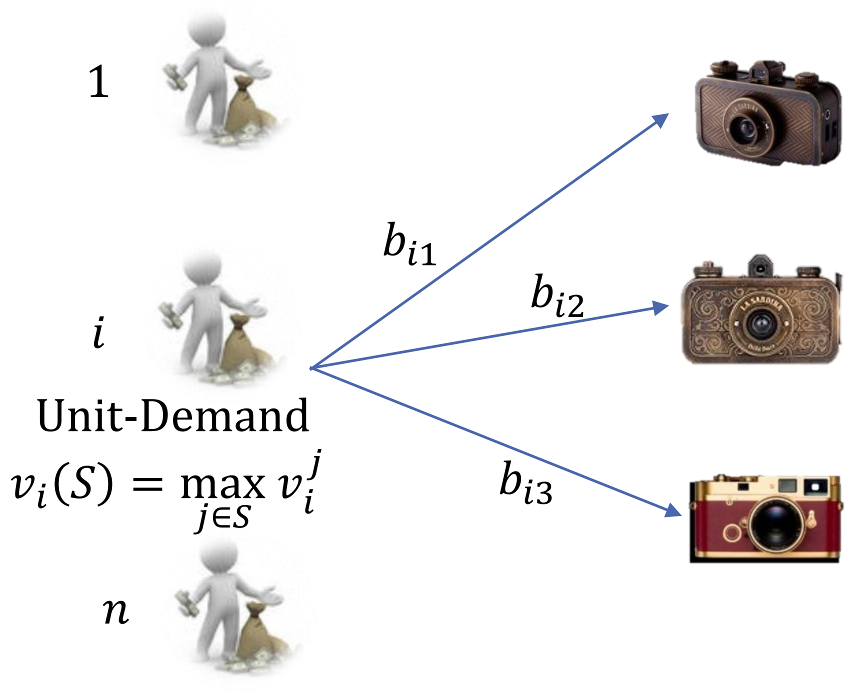
Proof.

$$\sum_i E[u_i(b(\mathbf{v}))] \geq \sum_i E\left[u_i\left(\frac{v_i}{2}, b_{-i}(v_{-i})\right)\right]$$

$$E[UTIL(b(\mathbf{v}))] + E[REV(b(\mathbf{v}))] \geq \frac{1}{2} \cdot E[OPT(\mathbf{v})]$$

$$E[SW(\mathrm{b}(\mathbf{v}))] \geq \frac{1}{2} \cdot E[OPT(\mathbf{v})]$$

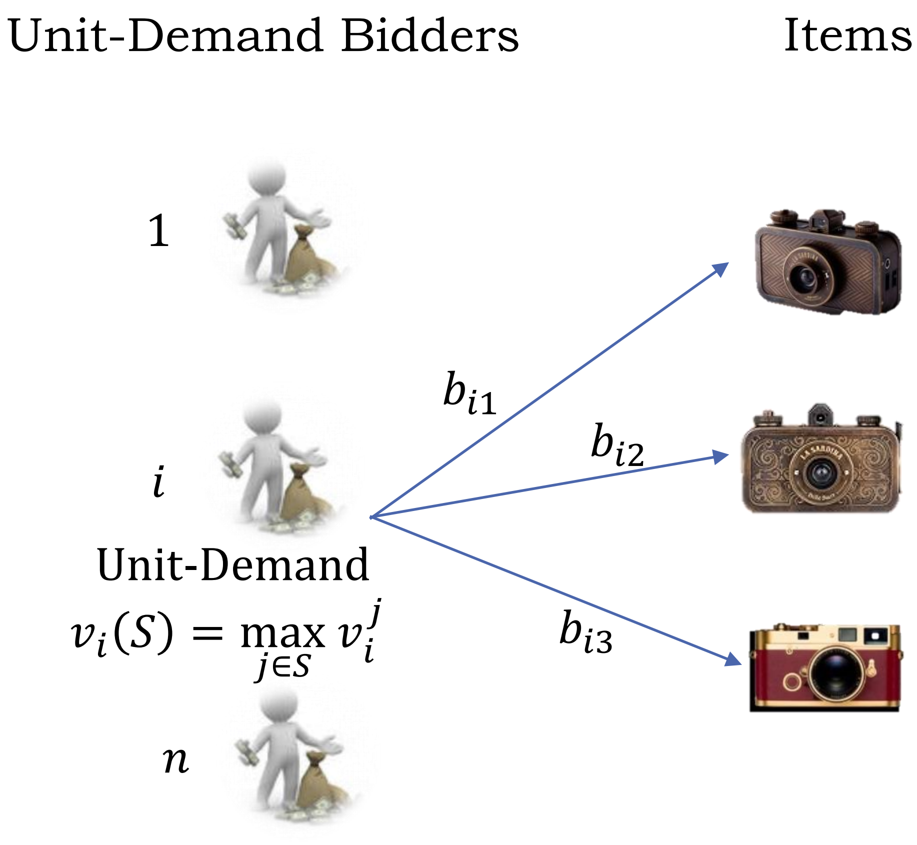
Matching Markets – Greedy Mechanism



- Allocated items greedily to highest remaining bid
- If allocated item $j(b)$, charge $b_i^{j(b)}$
- Utility:

$$u_i(b) = v_i^{j(b)} - b_i^{j(b)}$$

Matching Markets – Greedy Mechanism



- Deviation

$$b_i^j = \frac{v_i^j}{2}$$

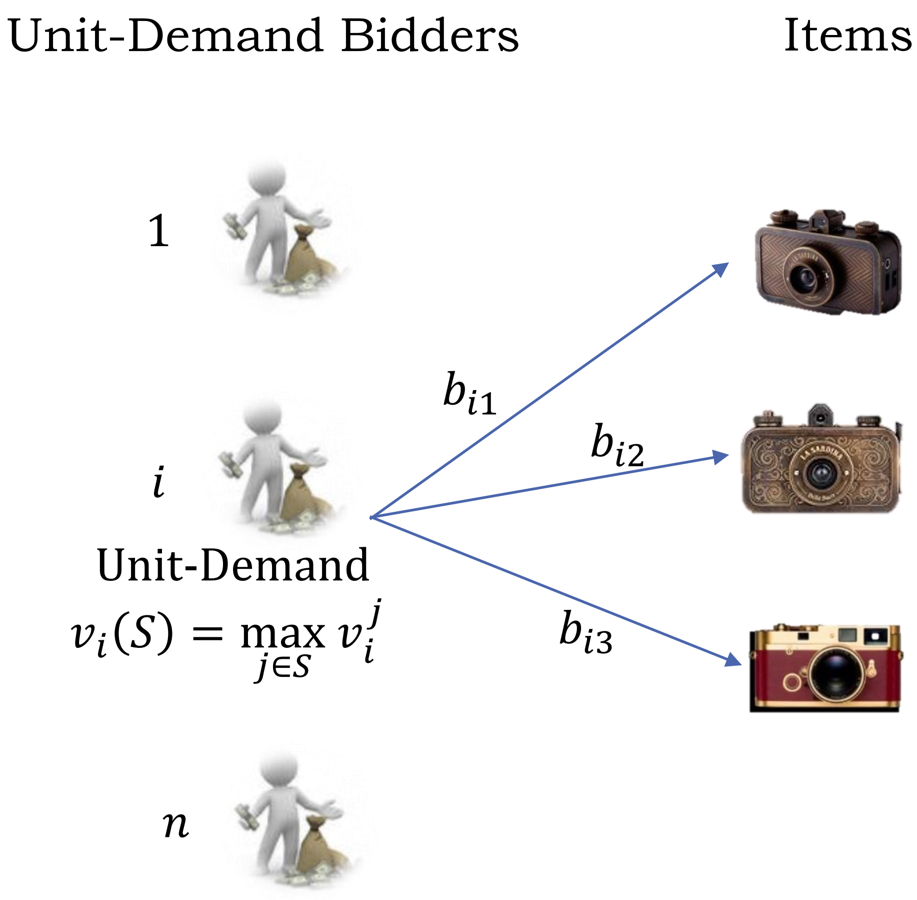
- Only for j =item in optimal matching

- If $p_j(b)$ is price of item j

$$u_i(b_i', b_{-i}) \geq \frac{v_i^j}{2} - p_j(b)$$

- Thus $\left(\frac{1}{2}, 1\right)$-smooth via valuation profile dependent deviations

Matching Markets – Greedy Mechanism



Unit-Demand

$v_i(S) = \max_{j \in S} v_i^j$

- In fact

$$b_i^j \sim H\left(v_i^j\right)$$

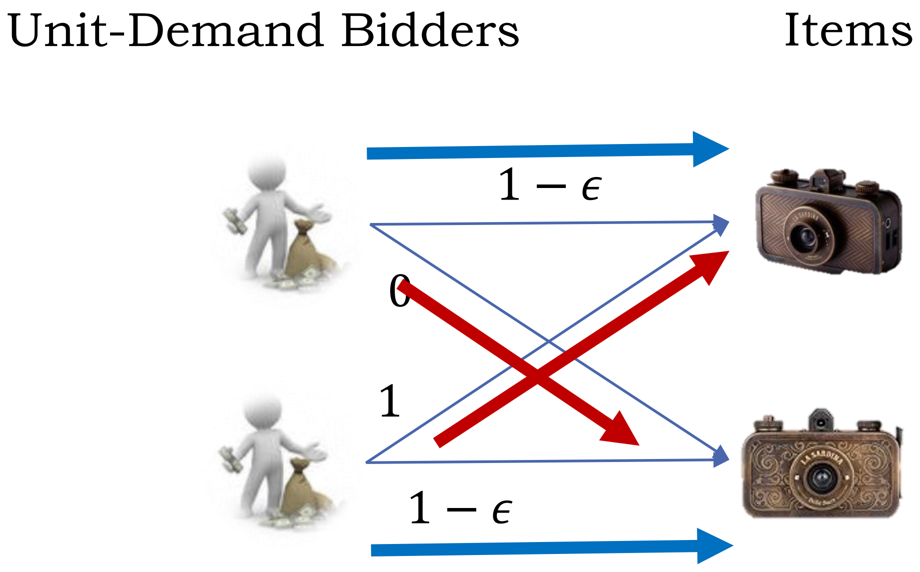
- Only for j =item in optimal matching

$$u_i(b_i', b_{-i}) \geq \left(1 - \frac{1}{e}\right) v_i^j - p_j(b)$$

- Thus $\left(1 - \frac{1}{e}, 1\right)$-smooth

- Greedy on true values: 2-approx.
- Greedy on reported values: 1.58-approx.!

Incentives improve algorithmic approximation

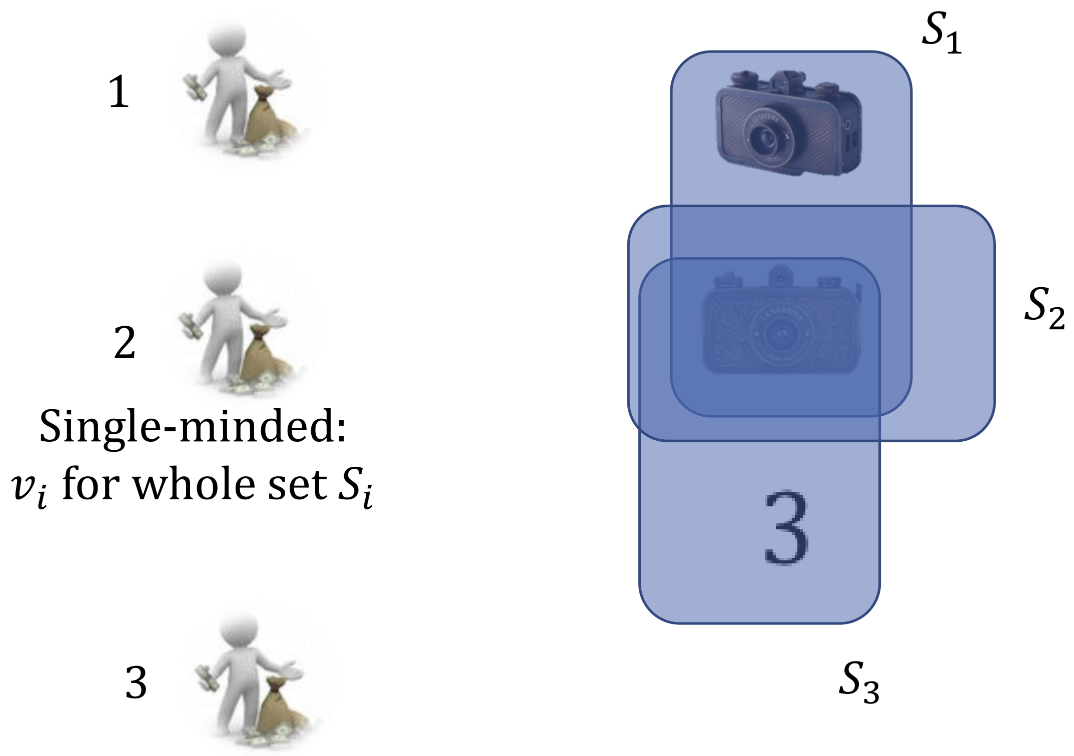


- Greedy on true values: 2-approx.
- At equilibrium:
  - Player 2 never goes for first item
  - Too expensive
  - So allocation is efficient

Single-Minded Combinatorial Auction

Single-Minded Bidders

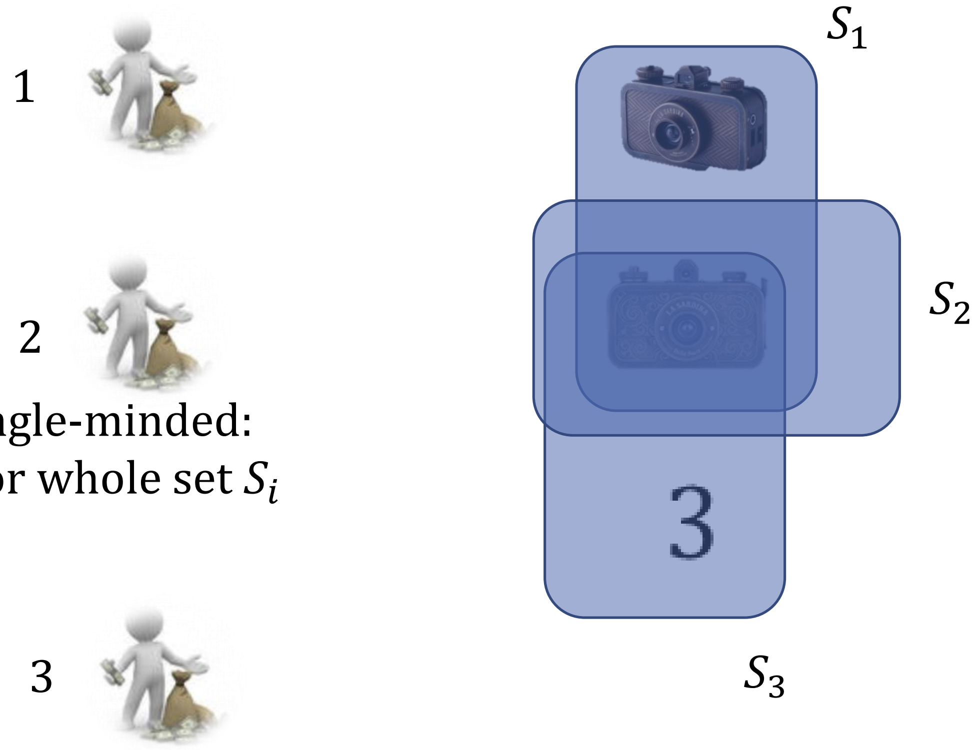
Items



- Each bidder submits b_i and T_i
- Run some algorithm over reported single-minded values
- Charge bid b_i if allocated

Optimal Algorithm

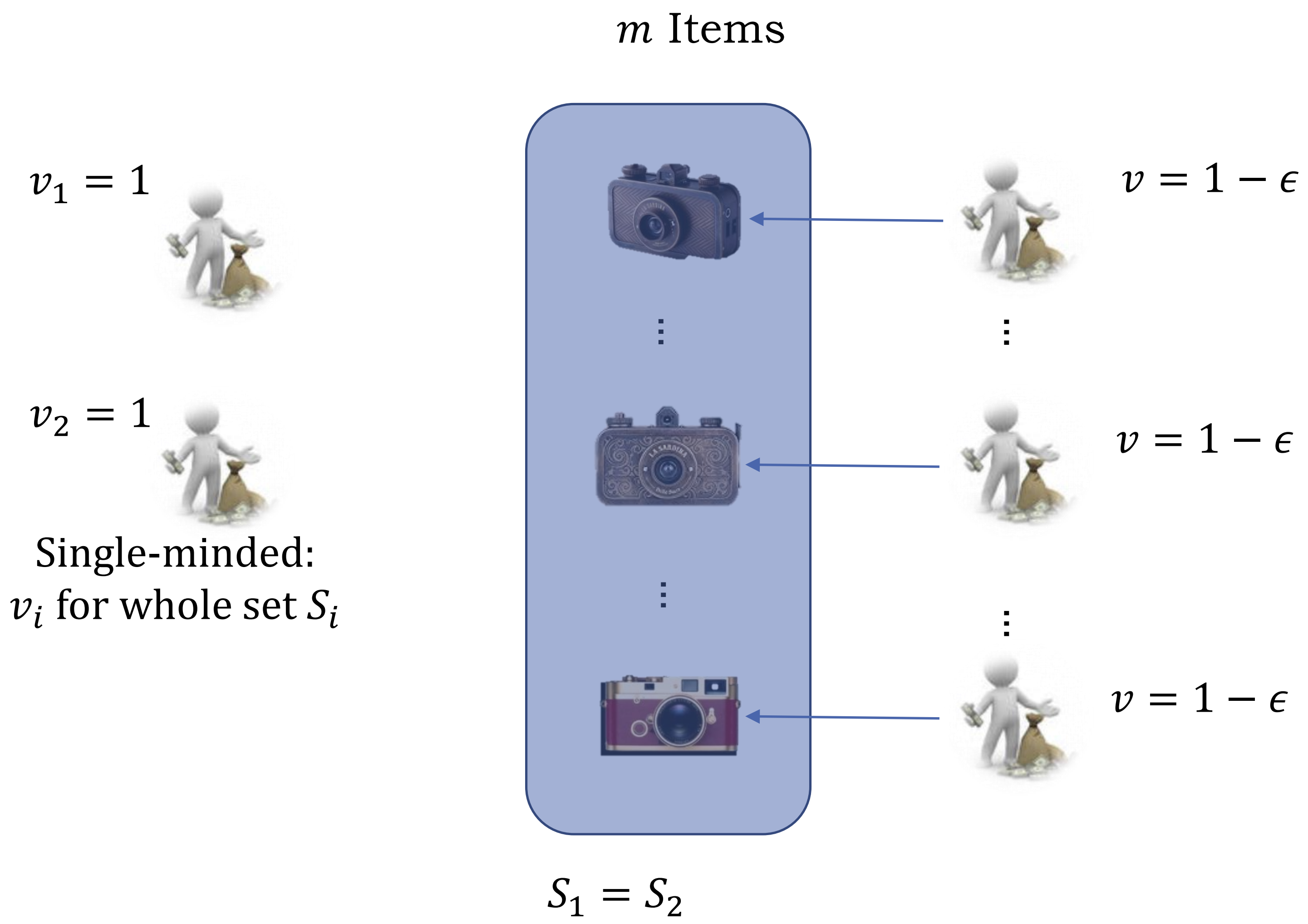
Single-Minded Bidders   Items



Single-minded: v_i for whole set S_i

- Each bidder submits b_i and T_i
- Run **optimal algorithm** over reported single-minded values
- Charge bid b_i if allocated

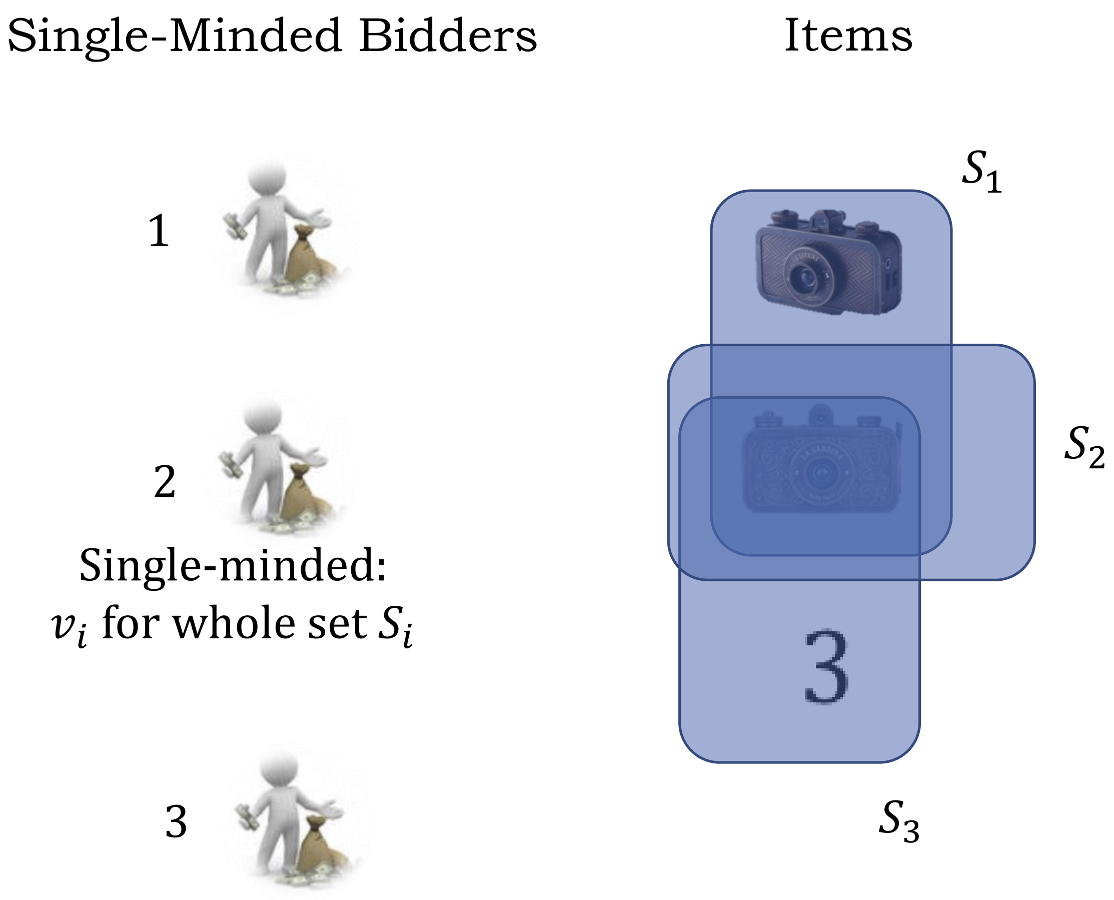
Linear inefficiency!



At equilibrium:

- 1 and 2 bid $b = 1$, $\mathrm{T} = [m]$
- Other players bid 0
- $SW = 1$ but $OPT = m$

$\sqrt{m}$ −Approximation Algorithm



- Each bidder submits b_i and T_i
- Run $\sqrt{m}$ **−Approximation Algorithm** over reported single-minded values
- Charge bid b_i if allocated

$\sqrt{m}$ –Approximation Algorithm

Single-Minded Bidders

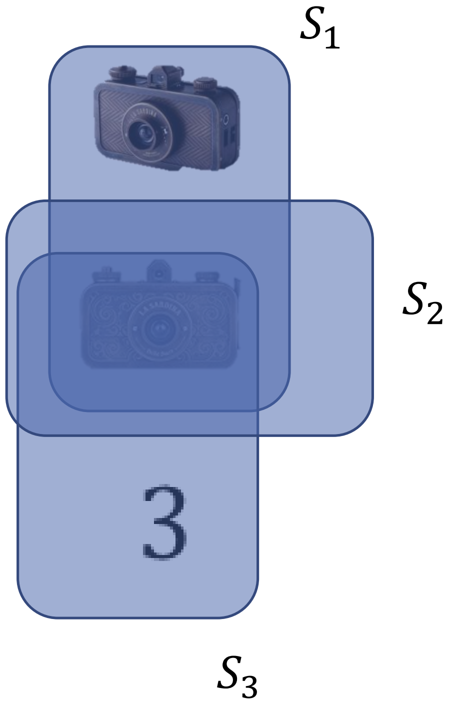
1 

2 

Single-minded:
v_i for whole set S_i

3 

Items

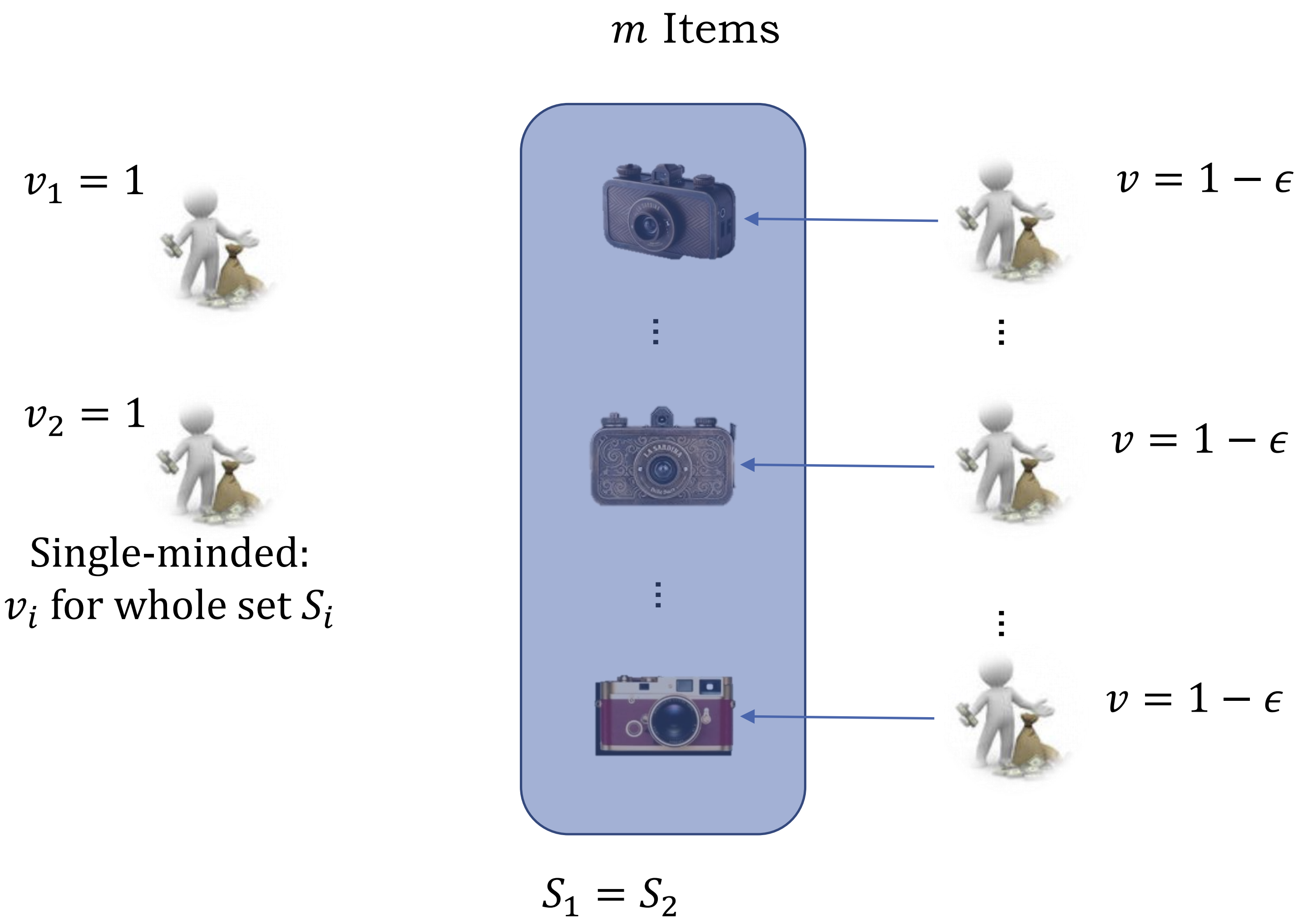


$\sqrt{m}$ –Approximation Algorithm

- Reweight bids as: $\boldsymbol{\hat{b}_i} = \frac{b_i}{\sqrt{|T_i|}}$
- Allocate in decreasing order of $\boldsymbol{\hat{b}_i}$
- Charge bid b_i if allocated

- Idea: A player can block at most $\sqrt{m}$ other players of same value from being allocated

Bad Example Corrected



Large players cannot block all small players

Smoothness of Approximation Algorithm

Single-Minded Bidders

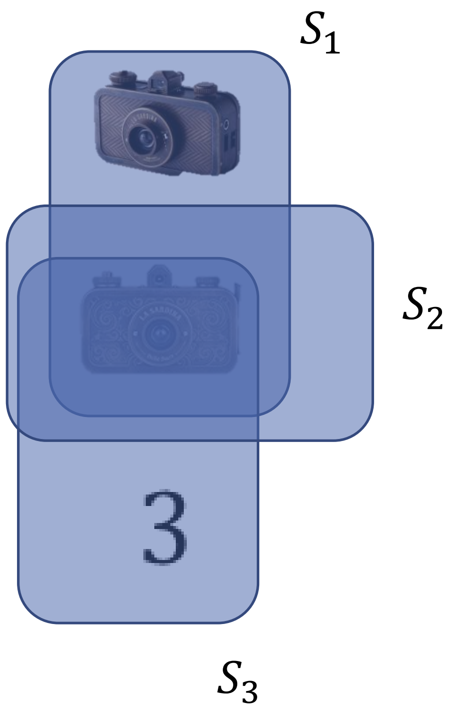
1 

2 

Single-minded:
v_i for whole set S_i

3 

Items



- Deviation b_i': bid $\frac{v_i}{2}$ for S_i
- Let τ_i(b): Threshold bid for being allocated S_i (including bid of player)
- By similar analysis:
$$u_i(b_i', b_{-i}) + \tau_i(b) \geq \frac{v_i}{2}$$
- Need to show: $\sum_i \tau_i(b) \leq c \cdot REV$

Smoothness of Approximation Algorithm

- Fact: Algorithm is $\sqrt{m}$ −approximation

- Think of hypothetical situation where each bidder is duplicated
- Duplicate bidder bids: $b_i = \tau_i(b) - \epsilon$ for set S_i

- By definition of $\tau_i(b)$: algorithm doesn't allocate to them
- Allocating to duplicate bidders yields welfare

$$\sum_i \tau_i(b)$$

- Since algorithm is $\sqrt{m}$ −approximation: $REV = \sum_i b_i X_i(b) \geq \frac{1}{\sqrt{m}} \sum_i \tau_i(b)$

Approximation improves efficiency

- Approximate mechanism: $\left(\frac{1}{2}, \sqrt{m}\right)$ −smooth

- Welfare at equilibrium $O(\sqrt{m})$-approximate NOT $O(m)$ −approximate

Some References

- **Smoothness**

  Roughgarden STOC'09, Lucier, Paes Leme EC'11, Roughgarden EC'12, Syrgkanis '12,

  Syrgkanis, Tardos STOC'13

- **Simultaneous First-Second Price Single-Item Auctions**

  Bikhchandani GEB'96, Christodoulou, Kovacs, Schapira ICALP'08, Bhawalkar, Roughgarden SODA'11, Hassidim, Kaplan, Mansour, Nisan EC'11, Feldman, Fu, Gravin, Lucier STOC'13

- **Auctions based on Greedy Allocation Algorithms**

  Lucier, Borodin SODA'10

- **AdAuctions (GSP, GFP)**

  Paes-Leme Tardos FOCS'10, Lucier, Paes-Leme + CKKK EC'11

- **Sequential First/Second Price Auctions**

  Paes Leme, Syrgkanis, Tardos SODA'12, Syrgkanis, Tardos EC'12

- **Multi-Unit Auctions**

  Bart de Keijzer et al. ESA'13

All above can be thought as smoothness proofs and some are compositions of auctions

This conference

Price of Anarchy in Auctions and Mechanisms

- Dutting, Henzinger, Stanberger. Valuation Compressions in VCG-Based Combinatorial Auctions
- Jose R. Correa, Andreas S. Schulz and Nicolas E. Stier-Moses. The Price of Anarchy of the Proportional Allocation Mechanism Revisited
- Jason Hartline, Darrell Hoy and Sam Taggart. Interim Smoothness for Auction Welfare and Revenue. (poster)
- Michal Feldman, Vasilis Syrgkanis and Brendan Lucier. Limits of Efficiency in Sequential Auctions
- Brendan Lucier, Yaron Singer, Vasilis Syrgkanis and Eva Tardos. Equilibrium in Combinatorial Public Projects

Price of Anarchy in Games

- Xinran He and David Kempe. Price of Anarchy for the N-player Competitive Cascade Game with Submodular Activation Functions
- Mona Rahn and Guido Schäfer. Bounding the Inefficiency of Altruism Through Social Contribution Games
- Yoram Bachrach, Vasilis Syrgkanis and Milan Vojnovic. Incentives and Efficiency in Uncertain Collaborative Environments